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# Preface

The purpose of this handbook is to supply a collection of mathematical formulas and tables which will prove to be valuable to students and research workers in the fields of mathematics, physics, engineering and other sciences. To accomplish this, care has been taken to include those formulas and tables which are most likely to be needed in practice rather than highly specialized results which are rarely used. Every effort has been made to present results concisely as well as precisely so that they may be referred to with a maximum of ease as well as confidence.

Topics covered range from elementary to advanced. Elementary topics include those from algebra, geometry, trigonometry, analytic geometry and calculus. Advanced topics include those from differential equations, vector analysis, Fourier series, gamma and beta functions, Bessel and Legendre functions, Fourier and Laplace transforms, elliptic functions and various other special functions of importance. This wide coverage of topics has been adopted so as to provide within a single volume most of the important mathematical results needed by the student or research worker regardless of his particular field of interest or level of attainment.

The book is divided into two main parts. Part I presents mathematical formulas together with other material, such as definitions, theorems, graphs, diagrams, etc., essential for proper understanding and application of the formulas. Included in this first part are extensive tables of integrals and Laplace transforms which should be extremely useful to the student and research worker. Part II presents numerical tables such as the values of elementary functions (trigonometric, logarithmic, exponential, hyperbolic, etc.) as well as advanced functions (Bessel, Legendre, elliptic, etc.). In order to eliminate confusion, especially to the beginner in mathematics, the numerical tables for each function are separated. Thus, for example, the sine and cosine functions for angles in degrees and minutes are given in separate tables rather than in one table so that there is no need to be concerned about the possibility of error due to looking in the wrong column or row.

I wish to thank the various authors and publishers who gave me permission to adapt data from their books for use in several tables of this handbook. Appropriate references to such sources are given next to the corresponding tables. In particular I am indebted to the Literary Executor of the late Sir Ronald A. Fisher, F.R.S., to Dr. Frank Yates, F.R.S., and to Oliver and Boyd Ltd., Edinburgh, for permission to use data from Table III of their book *Statistical Tables for Biological, Agricultural and Medical Research.*

I also wish to express my gratitude to Nicola Monti, Henry Hayden and Jack Margolin for their excellent editorial cooperation.

M. R. SPIEGEL

Rensselaer Polytechnic Institute
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# *Part I*

# FORMULAS

# THE GREEK ALPHABET

| Greek name | Greek letter | |
|---|---|---|
| | Lower case | Capital |
| Alpha | $\alpha$ | A |
| Beta | $\beta$ | B |
| Gamma | $\gamma$ | $\Gamma$ |
| Delta | $\delta$ | $\Delta$ |
| Epsilon | $\epsilon$ | E |
| Zeta | $\zeta$ | Z |
| Eta | $\eta$ | H |
| Theta | $\theta$ | $\Theta$ |
| Iota | $\iota$ | I |
| Kappa | $\kappa$ | K |
| Lambda | $\lambda$ | $\Lambda$ |
| Mu | $\mu$ | M |

| Greek name | Greek letter | |
|---|---|---|
| | Lower case | Capital |
| Nu | $\nu$ | N |
| Xi | $\xi$ | $\Xi$ |
| Omicron | o | O |
| Pi | $\pi$ | $\Pi$ |
| Rho | $\rho$ | P |
| Sigma | $\sigma$ | $\Sigma$ |
| Tau | $\tau$ | T |
| Upsilon | $\upsilon$ | $\Upsilon$ |
| Phi | $\phi$ | $\Phi$ |
| Chi | $\chi$ | X |
| Psi | $\psi$ | $\Psi$ |
| Omega | $\omega$ | $\Omega$ |

# 1 SPECIAL CONSTANTS

**1.1** $\pi$ = 3.14159 26535 89793 23846 2643 38327950 288419716939937510 5820974944 5923078164062862089

**1.2** $e$ = 2.71828 18284 59045 23536 0287... = $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$
= natural base of logarithms

**1.3** $\sqrt{2}$ = 1.41421 35623 73095 04880...

**1.4** $\sqrt{3}$ = 1.73205 08075 68877 2935...

**1.5** $\sqrt{5}$ = 2.23606 79774 99789 6964...

**1.6** $\sqrt[3]{2}$ = 1.25992 1050...

**1.7** $\sqrt[3]{3}$ = 1.44224 9570...

**1.8** $\sqrt[5]{2}$ = 1.14869 8355...

**1.9** $\sqrt[5]{3}$ = 1.24573 0940...

**1.10** $e^{\pi}$ = 23.14069 26327 79269 006...

**1.11** $\pi^{e}$ = 22.45915 77183 61045 47342 715...

**1.12** $e^{e}$ = 15.15426 22414 79264 190...

**1.13** $\log_{10} 2$ = 0.30102 99956 63981 19521 37389...

**1.14** $\log_{10} 3$ = 0.47712 12547 19662 43729 50279...

**1.15** $\log_{10} e$ = 0.43429 44819 03251 82765...

**1.16** $\log_{10} \pi$ = 0.49714 98726 94133 85435 12683...

**1.17** $\log_e 10$ = ln 10 = 2.30258 50929 94045 68401 7991...

**1.18** $\log_e 2$ = ln 2 = 0.69314 71805 59945 30941 7232...

**1.19** $\log_e 3$ = ln 3 = 1.09861 22886 68109 69139 5245...

**1.20** $\gamma$ = 0.57721 56649 01532 86060 6512... = *Euler's constant*

$$= \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\right)$$

**1.21** $e^{\gamma}$ = 1.78107 24179 90197 9852... [see 1.20]

**1.22** $\sqrt{e}$ = 1.64872 12707 00128 1468...

**1.23** $\sqrt{\pi} = \Gamma(\frac{1}{2})$ = 1.77245 38509 05516 02729 8167...
where $\Gamma$ is the *gamma function* [see pages 101-102].

**1.24** $\Gamma(\frac{1}{3})$ = 2.67893 85347 07748...

**1.25** $\Gamma(\frac{1}{4})$ = 3.62560 99082 21908...

**1.26** 1 radian = $180°/\pi$ = 57.29577 95130 8232...°

**1.27** 1° = $\pi/180$ radians = 0.01745 32925 19943 29576 92... radians

# 2 SPECIAL PRODUCTS and FACTORS

**2.1** $(x+y)^2 = x^2 + 2xy + y^2$

**2.2** $(x-y)^2 = x^2 - 2xy + y^2$

**2.3** $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

**2.4** $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

**2.5** $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

**2.6** $(x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

**2.7** $(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

**2.8** $(x-y)^5 = x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$

**2.9** $(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

**2.10** $(x-y)^6 = x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6$

The results 2.1 to 2.10 above are special cases of the *binomial formula* [see page 3].

**2.11** $x^2 - y^2 = (x-y)(x+y)$

**2.12** $x^3 - y^3 = (x-y)(x^2+xy+y^2)$

**2.13** $x^3 + y^3 = (x+y)(x^2-xy+y^2)$

**2.14** $x^4 - y^4 = (x-y)(x+y)(x^2+y^2)$

**2.15** $x^5 - y^5 = (x-y)(x^4+x^3y+x^2y^2+xy^3+y^4)$

**2.16** $x^5 + y^5 = (x+y)(x^4-x^3y+x^2y^2-xy^3+y^4)$

**2.17** $x^6 - y^6 = (x-y)(x+y)(x^2+xy+y^2)(x^2-xy+y^2)$

**2.18** $x^4 + x^2y^2 + y^4 = (x^2+xy+y^2)(x^2-xy+y^2)$

**2.19** $x^4 + 4y^4 = (x^2+2xy+2y^2)(x^2-2xy+2y^2)$

Some generalizations of the above are given by the following results where $n$ is a positive integer.

**2.20**

$$\begin{aligned} x^{2n+1} - y^{2n+1} &= (x-y)(x^{2n} + x^{2n-1}y + x^{2n-2}y^2 + \cdots + y^{2n}) \\ &= (x-y)\left(x^2 - 2xy\cos\frac{2\pi}{2n+1} + y^2\right)\left(x^2 - 2xy\cos\frac{4\pi}{2n+1} + y^2\right) \\ &\qquad \cdots \left(x^2 - 2xy\cos\frac{2n\pi}{2n+1} + y^2\right) \end{aligned}$$

**2.21**

$$\begin{aligned} x^{2n+1} + y^{2n+1} &= (x+y)(x^{2n} - x^{2n-1}y + x^{2n-2}y^2 - \cdots + y^{2n}) \\ &= (x+y)\left(x^2 + 2xy\cos\frac{2\pi}{2n+1} + y^2\right)\left(x^2 + 2xy\cos\frac{4\pi}{2n+1} + y^2\right) \\ &\qquad \cdots \left(x^2 + 2xy\cos\frac{2n\pi}{2n+1} + y^2\right) \end{aligned}$$

**2.22**

$$\begin{aligned} x^{2n} - y^{2n} &= (x-y)(x+y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots) \\ &= (x-y)(x+y)\left(x^2 - 2xy\cos\frac{\pi}{n} + y^2\right)\left(x^2 - 2xy\cos\frac{2\pi}{n} + y^2\right) \\ &\qquad \cdots \left(x^2 - 2xy\cos\frac{(n-1)\pi}{n} + y^2\right) \end{aligned}$$

**2.23**

$$\begin{aligned} x^{2n} + y^{2n} &= \left(x^2 + 2xy\cos\frac{\pi}{2n} + y^2\right)\left(x^2 + 2xy\cos\frac{3\pi}{2n} + y^2\right) \\ &\qquad \cdots \left(x^2 + 2xy\cos\frac{(2n-1)\pi}{2n} + y^2\right) \end{aligned}$$

# 3 The BINOMIAL FORMULA and BINOMIAL COEFFICIENTS

## FACTORIAL $n$

If $n = 1, 2, 3, \ldots$ *factorial* $n$ or $n$ *factorial* is defined as

**3.1** $$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$$

We also define *zero factorial* as

**3.2** $$0! = 1$$

## BINOMIAL FORMULA FOR POSITIVE INTEGRAL $n$

If $n = 1, 2, 3, \ldots$ then

**3.3** $$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots + y^n$$

This is called the *binomial formula.* It can be extended to other values of $n$ and then is an infinite series [see *Binomial Series,* page 110].

## BINOMIAL COEFFICIENTS

The result 3.3 can also be written

**3.4** $$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots + \binom{n}{n}y^n$$

where the coefficients, called *binomial coefficients,* are given by

**3.5** $$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{n-k}$$

## PROPERTIES OF BINOMIAL COEFFICIENTS

**3.6** $$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

This leads to *Pascal's triangle* [see page 236].

**3.7** $$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

**3.8** $$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots (-1)^n \binom{n}{n} = 0$$

**3.9** $$\binom{n}{n} + \binom{n+1}{n} + \binom{n+2}{n} + \cdots + \binom{n+m}{n} = \binom{n+m+1}{n+1}$$

**3.10** $$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1}$$

**3.11** $$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}$$

**3.12** $$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

**3.13** $$\binom{m}{0}\binom{n}{p} + \binom{m}{1}\binom{n}{p-1} + \cdots + \binom{m}{p}\binom{n}{0} = \binom{m+n}{p}$$

**3.14** $$(1)\binom{n}{1} + (2)\binom{n}{2} + (3)\binom{n}{3} + \cdots + (n)\binom{n}{n} = n2^{n-1}$$

**3.15** $$(1)\binom{n}{1} - (2)\binom{n}{2} + (3)\binom{n}{3} - \cdots (-1)^{n+1}(n)\binom{n}{n} = 0$$

## MULTINOMIAL FORMULA

**3.16** $$(x_1 + x_2 + \cdots + x_p)^n = \sum \frac{n!}{n_1!\, n_2! \cdots n_p!} x_1^{n_1} x_2^{n_2} \cdots x_p^{n_p}$$

where the sum, denoted by $\Sigma$, is taken over all nonnegative integers $n_1, n_2, \ldots, n_p$ for which $n_1 + n_2 + \cdots + n_p = n$.

# 4 GEOMETRIC FORMULAS

## RECTANGLE OF LENGTH $b$ AND WIDTH $a$

**4.1** Area $= ab$

**4.2** Perimeter $= 2a + 2b$

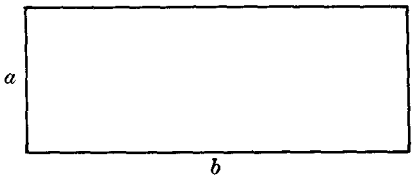


Fig. 4-1

## PARALLELOGRAM OF ALTITUDE $h$ AND BASE $b$

**4.3** Area $= bh = ab \sin \theta$

**4.4** Perimeter $= 2a + 2b$

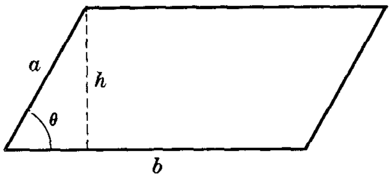


Fig. 4-2

## TRIANGLE OF ALTITUDE $h$ AND BASE $b$

**4.5** Area $= \frac{1}{2}bh = \frac{1}{2}ab \sin \theta$

$= \sqrt{s(s-a)(s-b)(s-c)}$

where $s = \frac{1}{2}(a+b+c) =$ semiperimeter

**4.6** Perimeter $= a + b + c$

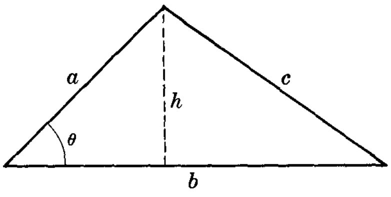


Fig. 4-3

## TRAPEZOID OF ALTITUDE $h$ AND PARALLEL SIDES $a$ AND $b$

**4.7** Area $= \frac{1}{2}h(a+b)$

**4.8** Perimeter $= a + b + h\left(\frac{1}{\sin \theta} + \frac{1}{\sin \phi}\right)$

$= a + b + h(\csc \theta + \csc \phi)$

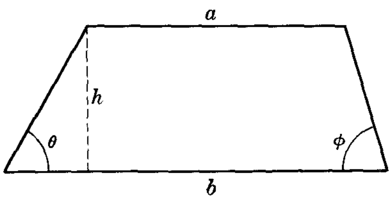


Fig. 4-4

## REGULAR POLYGON OF $n$ SIDES EACH OF LENGTH $b$

**4.9** Area $= \frac{1}{4}nb^2 \cot \frac{\pi}{n} = \frac{1}{4}nb^2 \frac{\cos(\pi/n)}{\sin(\pi/n)}$

**4.10** Perimeter $= nb$

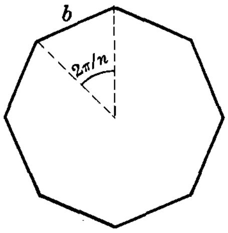


Fig. 4-5

## CIRCLE OF RADIUS $r$

**4.11** Area $= \pi r^2$

**4.12** Perimeter $= 2\pi r$

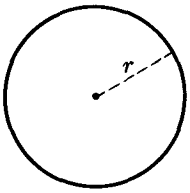


Fig. 4-6

## SECTOR OF CIRCLE OF RADIUS $r$

**4.13** Area $= \frac{1}{2}r^2\theta$ [$\theta$ in radians]

**4.14** Arc length $s = r\theta$

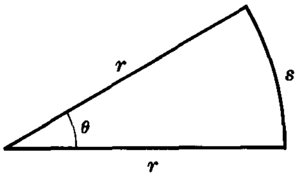


Fig. 4-7

## RADIUS OF CIRCLE INSCRIBED IN A TRIANGLE OF SIDES $a, b, c$

**4.15**
$$r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$$

where $s = \frac{1}{2}(a+b+c)$ = semiperimeter

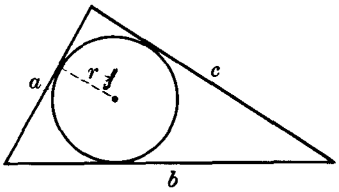


Fig. 4-8

## RADIUS OF CIRCLE CIRCUMSCRIBING A TRIANGLE OF SIDES $a, b, c$

**4.16**
$$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

where $s = \frac{1}{2}(a+b+c)$ = semiperimeter

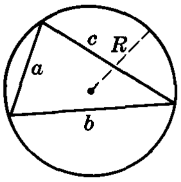


Fig. 4-9

## REGULAR POLYGON OF $n$ SIDES INSCRIBED IN CIRCLE OF RADIUS $r$

**4.17** Area $= \tfrac{1}{2}nr^2 \sin\frac{2\pi}{n} = \tfrac{1}{2}nr^2 \sin\frac{360^\circ}{n}$

**4.18** Perimeter $= 2nr \sin\frac{\pi}{n} = 2nr \sin\frac{180^\circ}{n}$

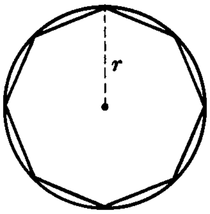


Fig. 4-10

## REGULAR POLYGON OF $n$ SIDES CIRCUMSCRIBING A CIRCLE OF RADIUS $r$

**4.19** Area $= nr^2 \tan\frac{\pi}{n} = nr^2 \tan\frac{180^\circ}{n}$

**4.20** Perimeter $= 2nr \tan\frac{\pi}{n} = 2nr \tan\frac{180^\circ}{n}$

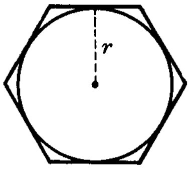


Fig. 4-11

## SEGMENT OF CIRCLE OF RADIUS $r$

**4.21** Area of shaded part $= \tfrac{1}{2}r^2(\theta - \sin\theta)$

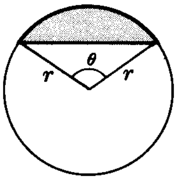


Fig. 4-12

## ELLIPSE OF SEMI-MAJOR AXIS $a$ AND SEMI-MINOR AXIS $b$

**4.22** Area $= \pi ab$

**4.23** Perimeter $= 4a\int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}\,d\theta$

$= 2\pi\sqrt{\tfrac{1}{2}(a^2+b^2)}$ [approximately]

where $k = \sqrt{a^2-b^2}/a$. See page 254 for numerical tables.

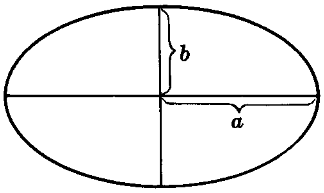


Fig. 4-13

## SEGMENT OF A PARABOLA

**4.24** Area $= \tfrac{2}{3}ab$

**4.25** Arc length $ABC = \tfrac{1}{2}\sqrt{b^2+16a^2} + \frac{b^2}{8a}\ln\left(\frac{4a+\sqrt{b^2+16a^2}}{b}\right)$

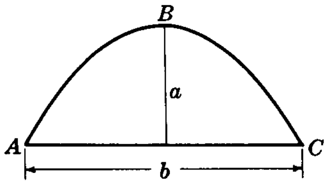


Fig. 4-14

## RECTANGULAR PARALLELEPIPED OF LENGTH $a$, HEIGHT $l$, WIDTH $c$

**4.26** Volume $= abc$

**4.27** Surface area $= 2(ab + ac + bc)$

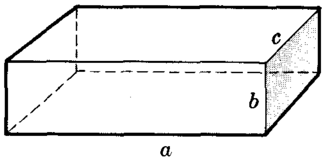


Fig. 4-15

## PARALLELEPIPED OF CROSS-SECTIONAL AREA $A$ AND HEIGHT $h$

**4.28** Volume $= Ah = abc \sin\theta$

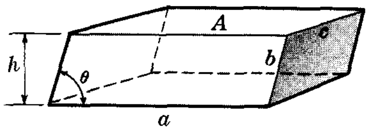


Fig. 4-16

## SPHERE OF RADIUS $r$

**4.29** Volume $= \frac{4}{3}\pi r^3$

**4.30** Surface area $= 4\pi r^2$

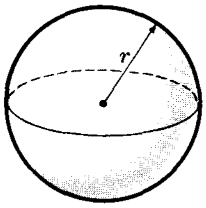


Fig. 4-17

## RIGHT CIRCULAR CYLINDER OF RADIUS $r$ AND HEIGHT $h$

**4.31** Volume $= \pi r^2 h$

**4.32** Lateral surface area $= 2\pi rh$

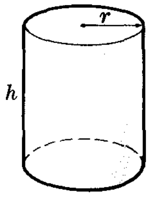


Fig. 4-18

## CIRCULAR CYLINDER OF RADIUS $r$ AND SLANT HEIGHT $l$

**4.33** Volume $= \pi r^2 h = \pi r^2 l \sin\theta$

**4.34** Lateral surface area $= 2\pi rl = \frac{2\pi rh}{\sin\theta} = 2\pi rh \csc\theta$

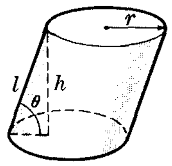


Fig. 4-19

## CYLINDER OF CROSS-SECTIONAL AREA $A$ AND SLANT HEIGHT $l$

**4.35** Volume $= Ah = Al \sin\theta$

**4.36** Lateral surface area $= pl = \dfrac{ph}{\sin\theta} = ph \csc\theta$

Note that formulas 4.31 to 4.34 are special cases.

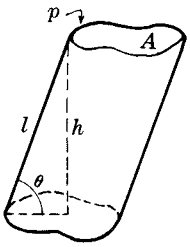


Fig. 4-20

## RIGHT CIRCULAR CONE OF RADIUS $r$ AND HEIGHT $h$

**4.37** Volume $= \frac{1}{3}\pi r^2 h$

**4.38** Lateral surface area $= \pi r\sqrt{r^2 + h^2} = \pi r l$

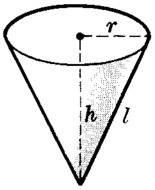


Fig. 4-21

## PYRAMID OF BASE AREA $A$ AND HEIGHT $h$

**4.39** Volume $= \frac{1}{3}Ah$

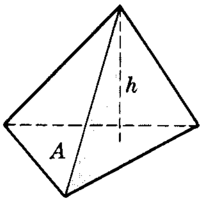


Fig. 4-22

## SPHERICAL CAP OF RADIUS $r$ AND HEIGHT $h$

**4.40** Volume (shaded in figure) $= \frac{1}{3}\pi h^2(3r - h)$

**4.41** Surface area $= 2\pi r h$

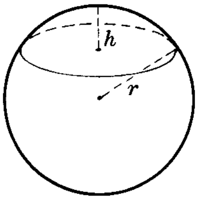


Fig. 4-23

## FRUSTRUM OF RIGHT CIRCULAR CONE OF RADII $a, b$ AND HEIGHT $h$

**4.42** Volume $= \frac{1}{3}\pi h(a^2 + ab + b^2)$

**4.43** Lateral surface area $= \pi(a + b)\sqrt{h^2 + (b - a)^2}$

$= \pi(a + b)l$

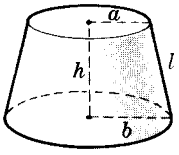


Fig. 4-24

## SPHERICAL TRIANGLE OF ANGLES $A, B, C$ ON SPHERE OF RADIUS $r$

**4.44** Area of triangle $ABC$ $= (A + B + C - \pi)r^2$

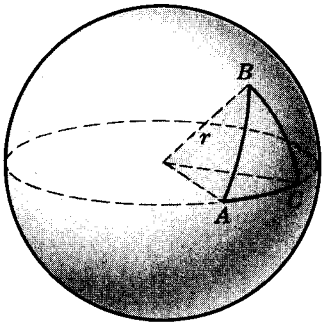


Fig. 4-25

## TORUS OF INNER RADIUS $a$ AND OUTER RADIUS $b$

**4.45** Volume $= \frac{1}{4}\pi^2(a + b)(b - a)^2$

**4.46** Surface area $= \pi^2(b^2 - a^2)$

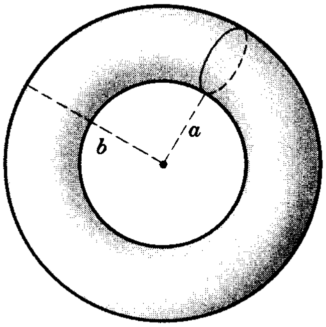


Fig. 4-26

## ELLIPSOID OF SEMI-AXES $a, b, c$

**4.47** Volume $= \frac{4}{3}\pi abc$

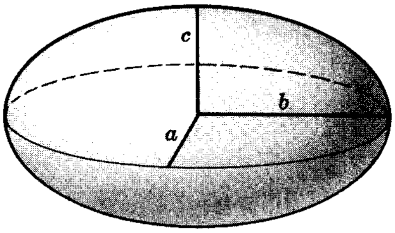


Fig. 4-27

## PARABOLOID OF REVOLUTION

**4.48** Volume $= \frac{1}{2}\pi b^2 a$

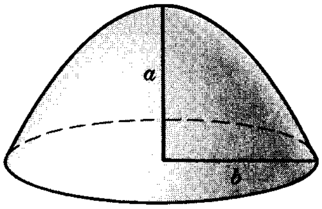


Fig. 4-28

# 5 TRIGONOMETRIC FUNCTIONS

## DEFINITION OF TRIGONOMETRIC FUNCTIONS FOR A RIGHT TRIANGLE

Triangle $ABC$ has a right angle (90°) at $C$ and sides of length $a, b, c$. The trigonometric functions of angle $A$ are defined as follows.

**5.1** $\quad$ *sine* of $A = \sin A = \dfrac{a}{c} = \dfrac{\text{opposite}}{\text{hypotenuse}}$

**5.2** $\quad$ *cosine* of $A = \cos A = \dfrac{b}{c} = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

**5.3** $\quad$ *tangent* of $A = \tan A = \dfrac{a}{b} = \dfrac{\text{opposite}}{\text{adjacent}}$

**5.4** $\quad$ *cotangent* of $A = \cot A = \dfrac{b}{a} = \dfrac{\text{adjacent}}{\text{opposite}}$

**5.5** $\quad$ *secant* of $A = \sec A = \dfrac{c}{b} = \dfrac{\text{hypotenuse}}{\text{adjacent}}$

**5.6** $\quad$ *cosecant* of $A = \csc A = \dfrac{c}{a} = \dfrac{\text{hypotenuse}}{\text{opposite}}$

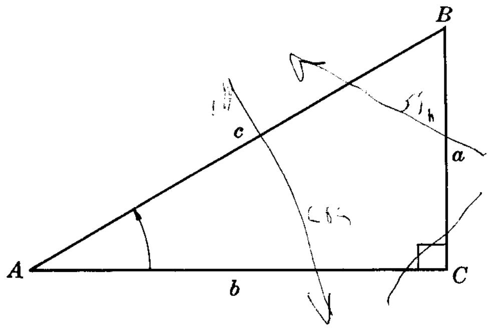


Fig. 5-1

## EXTENSIONS TO ANGLES WHICH MAY BE GREATER THAN 90°

Consider an $xy$ coordinate system [see Fig. 5-2 and 5-3 below]. A point $P$ in the $xy$ plane has coordinates $(x, y)$ where $x$ is considered as positive along $OX$ and negative along $OX'$ while $y$ is positive along $OY$ and negative along $OY'$. The distance from origin $O$ to point $P$ is positive and denoted by $r = \sqrt{x^2 + y^2}$. The angle $A$ described *counterclockwise* from $OX$ is considered *positive*. If it is described *clockwise* from $OX$ it is considered *negative*. We call $X'OX$ and $Y'OY$ the $x$ and $y$ axis respectively.

The various quadrants are denoted by I, II, III and IV called the first, second, third and fourth quadrants respectively. In Fig. 5-2, for example, angle $A$ is in the second quadrant while in Fig. 5-3 angle $A$ is in the third quadrant.

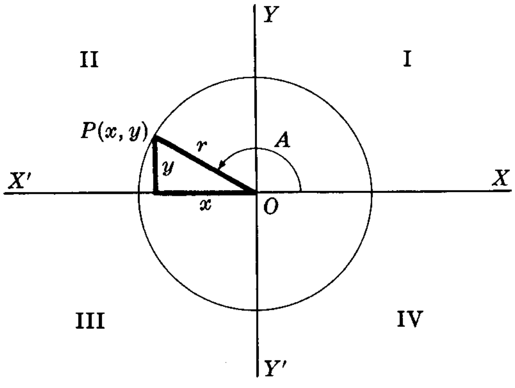


Fig. 5-2

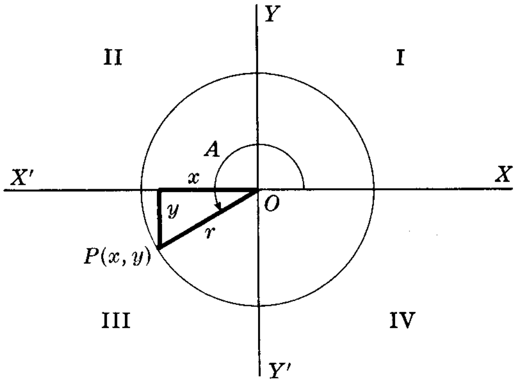


Fig. 5-3

For an angle $A$ in any quadrant the trigonometric functions of $A$ are defined as follows.

**5.7** $$\sin A = y/r$$

**5.8** $$\cos A = x/r$$

**5.9** $$\tan A = y/x$$

**5.10** $$\cot A = x/y$$

**5.11** $$\sec A = r/x$$

**5.12** $$\csc A = r/y$$

## RELATIONSHIP BETWEEN DEGREES AND RADIANS

A *radian* is that angle $\theta$ subtended at center $O$ of a circle by an arc $MN$ equal to the radius $r$.

Since $2\pi$ radians $= 360°$ we have

**5.13** $$1 \text{ radian} = 180°/\pi = 57.29577\ 95130\ 8232\ldots°$$

**5.14** $$1° = \pi/180 \text{ radians} = 0.01745\ 32925\ 19943\ 29576\ 92\ldots\text{radians}$$

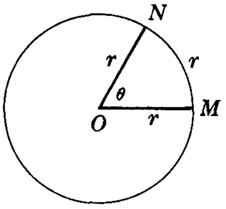


Fig. 5-4

## RELATIONSHIPS AMONG TRIGONOMETRIC FUNCTIONS

**5.15** $$\tan A = \frac{\sin A}{\cos A}$$

**5.16** $$\cot A = \frac{1}{\tan A} = \frac{\cos A}{\sin A}$$

**5.17** $$\sec A = \frac{1}{\cos A}$$

**5.18** $$\csc A = \frac{1}{\sin A}$$

**5.19** $$\sin^2 A + \cos^2 A = 1$$

**5.20** $$\sec^2 A - \tan^2 A = 1$$

**5.21** $$\csc^2 A - \cot^2 A = 1$$

## SIGNS AND VARIATIONS OF TRIGONOMETRIC FUNCTIONS

| Quadrant | $\sin A$ | $\cos A$ | $\tan A$ | $\cot A$ | $\sec A$ | $\csc A$ |
|---|---|---|---|---|---|---|
| I | $+$<br>0 to 1 | $+$<br>1 to 0 | $+$<br>0 to $\infty$ | $+$<br>$\infty$ to 0 | $+$<br>1 to $\infty$ | $+$<br>$\infty$ to 1 |
| II | $+$<br>1 to 0 | $-$<br>0 to $-1$ | $-$<br>$-\infty$ to 0 | $-$<br>0 to $-\infty$ | $-$<br>$-\infty$ to $-1$ | $+$<br>1 to $\infty$ |
| III | $-$<br>0 to $-1$ | $-$<br>$-1$ to 0 | $+$<br>0 to $\infty$ | $+$<br>$\infty$ to 0 | $-$<br>$-1$ to $-\infty$ | $-$<br>$-\infty$ to $-1$ |
| IV | $-$<br>$-1$ to 0 | $+$<br>0 to 1 | $-$<br>$-\infty$ to 0 | $-$<br>0 to $-\infty$ | $+$<br>$\infty$ to 1 | $-$<br>$-1$ to $-\infty$ |

## EXACT VALUES FOR TRIGONOMETRIC FUNCTIONS OF VARIOUS ANGLES

| Angle $A$ in degrees | Angle $A$ in radians | $\sin A$ | $\cos A$ | $\tan A$ | $\cot A$ | $\sec A$ | $\csc A$ |
|---|---|---|---|---|---|---|---|
| 0° | $0$ | $0$ | $1$ | $0$ | $\infty$ | $1$ | $\infty$ |
| 15° | $\pi/12$ | $\frac{1}{4}(\sqrt{6}-\sqrt{2})$ | $\frac{1}{4}(\sqrt{6}+\sqrt{2})$ | $2-\sqrt{3}$ | $2+\sqrt{3}$ | $\sqrt{6}-\sqrt{2}$ | $\sqrt{6}+\sqrt{2}$ |
| 30° | $\pi/6$ | $\frac{1}{2}$ | $\frac{1}{2}\sqrt{3}$ | $\frac{1}{3}\sqrt{3}$ | $\sqrt{3}$ | $\frac{2}{3}\sqrt{3}$ | $2$ |
| 45° | $\pi/4$ | $\frac{1}{2}\sqrt{2}$ | $\frac{1}{2}\sqrt{2}$ | $1$ | $1$ | $\sqrt{2}$ | $\sqrt{2}$ |
| 60° | $\pi/3$ | $\frac{1}{2}\sqrt{3}$ | $\frac{1}{2}$ | $\sqrt{3}$ | $\frac{1}{3}\sqrt{3}$ | $2$ | $\frac{2}{3}\sqrt{3}$ |
| 75° | $5\pi/12$ | $\frac{1}{4}(\sqrt{6}+\sqrt{2})$ | $\frac{1}{4}(\sqrt{6}-\sqrt{2})$ | $2+\sqrt{3}$ | $2-\sqrt{3}$ | $\sqrt{6}+\sqrt{2}$ | $\sqrt{6}-\sqrt{2}$ |
| 90° | $\pi/2$ | $1$ | $0$ | $\pm\infty$ | $0$ | $\pm\infty$ | $1$ |
| 105° | $7\pi/12$ | $\frac{1}{4}(\sqrt{6}+\sqrt{2})$ | $-\frac{1}{4}(\sqrt{6}-\sqrt{2})$ | $-(2+\sqrt{3})$ | $-(2-\sqrt{3})$ | $-(\sqrt{6}+\sqrt{2})$ | $\sqrt{6}-\sqrt{2}$ |
| 120° | $2\pi/3$ | $\frac{1}{2}\sqrt{3}$ | $-\frac{1}{2}$ | $-\sqrt{3}$ | $-\frac{1}{3}\sqrt{3}$ | $-2$ | $\frac{2}{3}\sqrt{3}$ |
| 135° | $3\pi/4$ | $\frac{1}{2}\sqrt{2}$ | $-\frac{1}{2}\sqrt{2}$ | $-1$ | $-1$ | $-\sqrt{2}$ | $\sqrt{2}$ |
| 150° | $5\pi/6$ | $\frac{1}{2}$ | $-\frac{1}{2}\sqrt{3}$ | $-\frac{1}{3}\sqrt{3}$ | $-\sqrt{3}$ | $-\frac{2}{3}\sqrt{3}$ | $2$ |
| 165° | $11\pi/12$ | $\frac{1}{4}(\sqrt{6}-\sqrt{2})$ | $-\frac{1}{4}(\sqrt{6}+\sqrt{2})$ | $-(2-\sqrt{3})$ | $-(2+\sqrt{3})$ | $-(\sqrt{6}-\sqrt{2})$ | $\sqrt{6}+\sqrt{2}$ |
| 180° | $\pi$ | $0$ | $-1$ | $0$ | $\mp\infty$ | $-1$ | $\pm\infty$ |
| 195° | $13\pi/12$ | $-\frac{1}{4}(\sqrt{6}-\sqrt{2})$ | $-\frac{1}{4}(\sqrt{6}+\sqrt{2})$ | $2-\sqrt{3}$ | $2+\sqrt{3}$ | $-(\sqrt{6}-\sqrt{2})$ | $-(\sqrt{6}+\sqrt{2})$ |
| 210° | $7\pi/6$ | $-\frac{1}{2}$ | $-\frac{1}{2}\sqrt{3}$ | $\frac{1}{3}\sqrt{3}$ | $\sqrt{3}$ | $-\frac{2}{3}\sqrt{3}$ | $-2$ |
| 225° | $5\pi/4$ | $-\frac{1}{2}\sqrt{2}$ | $-\frac{1}{2}\sqrt{2}$ | $1$ | $1$ | $-\sqrt{2}$ | $-\sqrt{2}$ |
| 240° | $4\pi/3$ | $-\frac{1}{2}\sqrt{3}$ | $-\frac{1}{2}$ | $\sqrt{3}$ | $\frac{1}{3}\sqrt{3}$ | $-2$ | $-\frac{2}{3}\sqrt{3}$ |
| 255° | $17\pi/12$ | $-\frac{1}{4}(\sqrt{6}+\sqrt{2})$ | $-\frac{1}{4}(\sqrt{6}-\sqrt{2})$ | $2+\sqrt{3}$ | $2-\sqrt{3}$ | $-(\sqrt{6}+\sqrt{2})$ | $-(\sqrt{6}-\sqrt{2})$ |
| 270° | $3\pi/2$ | $-1$ | $0$ | $\pm\infty$ | $0$ | $\mp\infty$ | $-1$ |
| 285° | $19\pi/12$ | $-\frac{1}{4}(\sqrt{6}+\sqrt{2})$ | $\frac{1}{4}(\sqrt{6}-\sqrt{2})$ | $-(2+\sqrt{3})$ | $-(2-\sqrt{3})$ | $\sqrt{6}+\sqrt{2}$ | $-(\sqrt{6}-\sqrt{2})$ |
| 300° | $5\pi/3$ | $-\frac{1}{2}\sqrt{3}$ | $\frac{1}{2}$ | $-\sqrt{3}$ | $-\frac{1}{3}\sqrt{3}$ | $2$ | $-\frac{2}{3}\sqrt{3}$ |
| 315° | $7\pi/4$ | $-\frac{1}{2}\sqrt{2}$ | $\frac{1}{2}\sqrt{2}$ | $-1$ | $-1$ | $\sqrt{2}$ | $-\sqrt{2}$ |
| 330° | $11\pi/6$ | $-\frac{1}{2}$ | $\frac{1}{2}\sqrt{3}$ | $-\frac{1}{3}\sqrt{3}$ | $-\sqrt{3}$ | $\frac{2}{3}\sqrt{3}$ | $-2$ |
| 345° | $23\pi/12$ | $-\frac{1}{4}(\sqrt{6}-\sqrt{2})$ | $\frac{1}{4}(\sqrt{6}+\sqrt{2})$ | $-(2-\sqrt{3})$ | $-(2+\sqrt{3})$ | $\sqrt{6}-\sqrt{2}$ | $-(\sqrt{6}+\sqrt{2})$ |
| 360° | $2\pi$ | $0$ | $1$ | $0$ | $\mp\infty$ | $1$ | $\mp\infty$ |

For tables involving other angles see pages 206-211 and 212-215.

## GRAPHS OF TRIGONOMETRIC FUNCTIONS

In each graph $x$ is in radians.

**5.22** $y = \sin x$

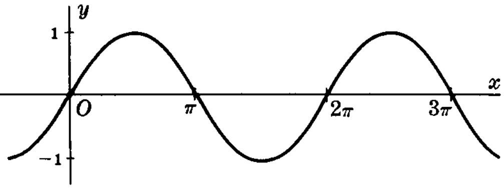


Fig. 5-5

**5.23** $y = \cos x$

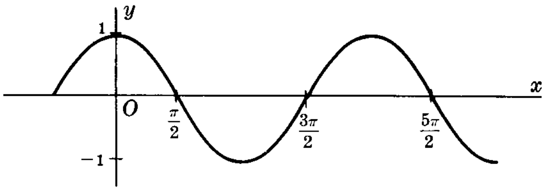


Fig. 5-6

**5.24** $y = \tan x$

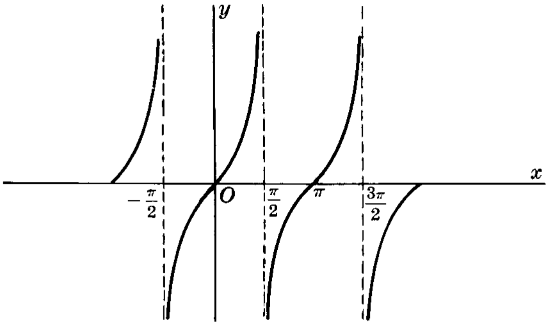


Fig. 5-7

**5.25** $y = \cot x$

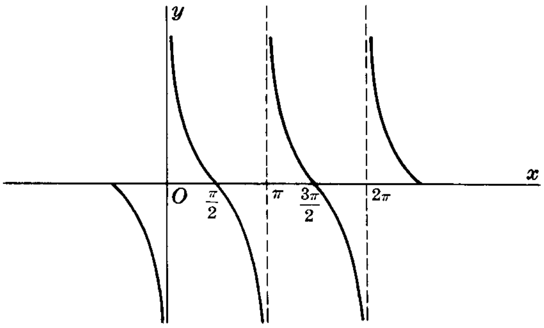


Fig. 5-8

**5.26** $y = \sec x$

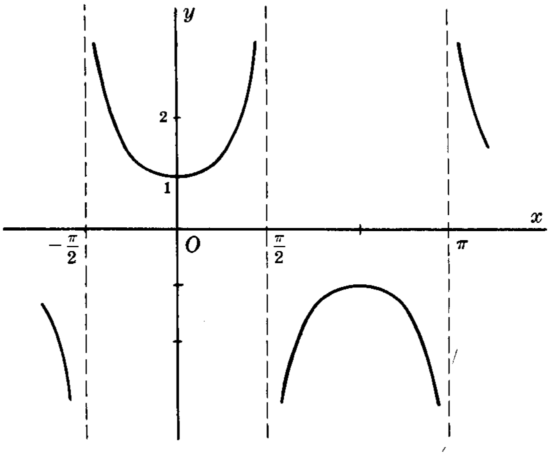


Fig. 5-9

**5.27** $y = \csc x$

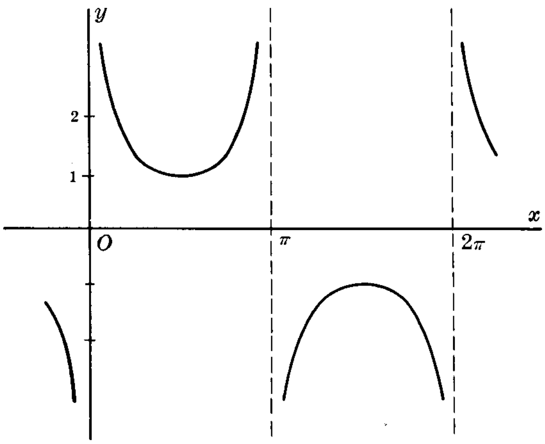


Fig. 5-10

## FUNCTIONS OF NEGATIVE ANGLES

**5.28** $\sin(-A) = -\sin A$ **5.29** $\cos(-A) = \cos A$ **5.30** $\tan(-A) = -\tan A$

**5.31** $\csc(-A) = -\csc A$ **5.32** $\sec(-A) = \sec A$ **5.33** $\cot(-A) = -\cot A$

## ADDITION FORMULAS

**5.34** $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

**5.35** $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

**5.36** $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

**5.37** $\cot(A \pm B) = \dfrac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$

## FUNCTIONS OF ANGLES IN ALL QUADRANTS IN TERMS OF THOSE IN QUADRANT I

| | $-A$ | $90^\circ \pm A$<br>$\frac{\pi}{2} \pm A$ | $180^\circ \pm A$<br>$\pi \pm A$ | $270^\circ \pm A$<br>$\frac{3\pi}{2} \pm A$ | $k(360^\circ) \pm A$<br>$2k\pi \pm A$<br>$k =$ integer |
|---|---|---|---|---|---|
| sin | $-\sin A$ | $\cos A$ | $\mp \sin A$ | $-\cos A$ | $\pm \sin A$ |
| cos | $\cos A$ | $\mp \sin A$ | $-\cos A$ | $\pm \sin A$ | $\cos A$ |
| tan | $-\tan A$ | $\mp \cot A$ | $\pm \tan A$ | $\mp \cot A$ | $\pm \tan A$ |
| csc | $-\csc A$ | $\sec A$ | $\mp \csc A$ | $-\sec A$ | $\pm \csc A$ |
| sec | $\sec A$ | $\mp \csc A$ | $-\sec A$ | $\pm \csc A$ | $\sec A$ |
| cot | $-\cot A$ | $\mp \tan A$ | $\pm \cot A$ | $\mp \tan A$ | $\pm \cot A$ |

## RELATIONSHIPS AMONG FUNCTIONS OF ANGLES IN QUADRANT I

| | $\sin A = u$ | $\cos A = u$ | $\tan A = u$ | $\cot A = u$ | $\sec A = u$ | $\csc A = u$ |
|---|---|---|---|---|---|---|
| $\sin A$ | $u$ | $\sqrt{1-u^2}$ | $u/\sqrt{1+u^2}$ | $1/\sqrt{1+u^2}$ | $\sqrt{u^2-1}/u$ | $1/u$ |
| $\cos A$ | $\sqrt{1-u^2}$ | $u$ | $1/\sqrt{1+u^2}$ | $u/\sqrt{1+u^2}$ | $1/u$ | $\sqrt{u^2-1}/u$ |
| $\tan A$ | $u/\sqrt{1-u^2}$ | $\sqrt{1-u^2}/u$ | $u$ | $1/u$ | $\sqrt{u^2-1}$ | $1/\sqrt{u^2-1}$ |
| $\cot A$ | $\sqrt{1-u^2}/u$ | $u/\sqrt{1-u^2}$ | $1/u$ | $u$ | $1/\sqrt{u^2-1}$ | $\sqrt{u^2-1}$ |
| $\sec A$ | $1/\sqrt{1-u^2}$ | $1/u$ | $\sqrt{1+u^2}$ | $\sqrt{1+u^2}/u$ | $u$ | $u/\sqrt{u^2-1}$ |
| $\csc A$ | $1/u$ | $1/\sqrt{1-u^2}$ | $\sqrt{1+u^2}/u$ | $\sqrt{1+u^2}$ | $u/\sqrt{u^2-1}$ | $\sqrt{1+u^2}$ |

For extensions to other quadrants use appropriate signs as given in the preceding table.

## DOUBLE ANGLE FORMULAS

**5.38** $\sin 2A = 2\sin A\cos A$

**5.39** $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$

**5.40** $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

## HALF ANGLE FORMULAS

**5.41** $\sin\dfrac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{2}} \quad \left[\begin{array}{l} + \text{ if } A/2 \text{ is in quadrant I or II} \\ - \text{ if } A/2 \text{ is in quadrant III or IV} \end{array}\right]$

**5.42** $\cos\dfrac{A}{2} = \pm\sqrt{\dfrac{1+\cos A}{2}} \quad \left[\begin{array}{l} + \text{ if } A/2 \text{ is in quadrant I or IV} \\ - \text{ if } A/2 \text{ is in quadrant II or III} \end{array}\right]$

**5.43** $\tan\dfrac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{1+\cos A}} \quad \left[\begin{array}{l} + \text{ if } A/2 \text{ is in quadrant I or III} \\ - \text{ if } A/2 \text{ is in quadrant II or IV} \end{array}\right]$

$= \dfrac{\sin A}{1+\cos A} = \dfrac{1-\cos A}{\sin A} = \csc A - \cot A$

## MULTIPLE ANGLE FORMULAS

**5.44** $\sin 3A = 3\sin A - 4\sin^3 A$

**5.45** $\cos 3A = 4\cos^3 A - 3\cos A$

**5.46** $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

**5.47** $\sin 4A = 4\sin A\cos A - 8\sin^3 A\cos A$

**5.48** $\cos 4A = 8\cos^4 A - 8\cos^2 A + 1$

**5.49** $\tan 4A = \dfrac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}$

**5.50** $\sin 5A = 5\sin A - 20\sin^3 A + 16\sin^5 A$

**5.51** $\cos 5A = 16\cos^5 A - 20\cos^3 A + 5\cos A$

**5.52** $\tan 5A = \dfrac{\tan^5 A - 10\tan^3 A + 5\tan A}{1 - 10\tan^2 A + 5\tan^4 A}$

See also formulas 5.68 and 5.69.

## POWERS OF TRIGONOMETRIC FUNCTIONS

**5.53** $\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A$

**5.54** $\cos^2 A = \frac{1}{2} + \frac{1}{2}\cos 2A$

**5.55** $\sin^3 A = \frac{3}{4}\sin A - \frac{1}{4}\sin 3A$

**5.56** $\cos^3 A = \frac{3}{4}\cos A + \frac{1}{4}\cos 3A$

**5.57** $\sin^4 A = \frac{3}{8} - \frac{1}{2}\cos 2A + \frac{1}{8}\cos 4A$

**5.58** $\cos^4 A = \frac{3}{8} + \frac{1}{2}\cos 2A + \frac{1}{8}\cos 4A$

**5.59** $\sin^5 A = \frac{5}{8}\sin A - \frac{5}{16}\sin 3A + \frac{1}{16}\sin 5A$

**5.60** $\cos^5 A = \frac{5}{8}\cos A + \frac{5}{16}\cos 3A + \frac{1}{16}\cos 5A$

See also formulas 5.70 through 5.73.

## SUM, DIFFERENCE AND PRODUCT OF TRIGONOMETRIC FUNCTIONS

**5.61** $$\sin A + \sin B = 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)$$

**5.62** $$\sin A - \sin B = 2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)$$

**5.63** $$\cos A + \cos B = 2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)$$

**5.64** $$\cos A - \cos B = 2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(B-A)$$

**5.65** $$\sin A\sin B = \tfrac{1}{2}\{\cos(A-B) - \cos(A+B)\}$$

**5.66** $$\cos A\cos B = \tfrac{1}{2}\{\cos(A-B) + \cos(A+B)\}$$

**5.67** $$\sin A\cos B = \tfrac{1}{2}\{\sin(A-B) + \sin(A+B)\}$$

## GENERAL FORMULAS

**5.68** $$\sin nA = \sin A\left\{(2\cos A)^{n-1} - \binom{n-2}{1}(2\cos A)^{n-3} + \binom{n-3}{2}(2\cos A)^{n-5} - \cdots\right\}$$

**5.69** $$\begin{aligned}\cos nA = \frac{1}{2}\Big\{&(2\cos A)^{n} - \frac{n}{1}(2\cos A)^{n-2} + \frac{n}{2}\binom{n-3}{1}(2\cos A)^{n-4}\\ &- \frac{n}{3}\binom{n-4}{2}(2\cos A)^{n-6} + \cdots\Big\}\end{aligned}$$

**5.70** $$\sin^{2n-1}A = \frac{(-1)^{n-1}}{2^{2n-2}}\left\{\sin(2n-1)A - \binom{2n-1}{1}\sin(2n-3)A + \cdots (-1)^{n-1}\binom{2n-1}{n-1}\sin A\right\}$$

**5.71** $$\cos^{2n-1}A = \frac{1}{2^{2n-2}}\left\{\cos(2n-1)A + \binom{2n-1}{1}\cos(2n-3)A + \cdots + \binom{2n-1}{n-1}\cos A\right\}$$

**5.72** $$\sin^{2n}A = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{(-1)^{n}}{2^{2n-1}}\left\{\cos 2nA - \binom{2n}{1}\cos(2n-2)A + \cdots (-1)^{n-1}\binom{2n}{n-1}\cos 2A\right\}$$

**5.73** $$\cos^{2n}A = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}}\left\{\cos 2nA + \binom{2n}{1}\cos(2n-2)A + \cdots + \binom{2n}{n-1}\cos 2A\right\}$$

## INVERSE TRIGONOMETRIC FUNCTIONS

If $x = \sin y$ then $y = \sin^{-1}x$, i.e. *the angle whose sine is $x$* or *inverse sine of $x$*, is a many-valued function of $x$ which is a collection of single-valued functions called *branches*. Similarly the other inverse trigonometric functions are multiple-valued.

For many purposes a particular branch is required. This is called the *principal branch* and the values for this branch are called *principal values*.

## PRINCIPAL VALUES FOR INVERSE TRIGONOMETRIC FUNCTIONS

| Principal values for $x \geqq 0$ | Principal values for $x < 0$ |
|---|---|
| $0 \leqq \sin^{-1} x \leqq \pi/2$ | $-\pi/2 \leqq \sin^{-1} x < 0$ |
| $0 \leqq \cos^{-1} x \leqq \pi/2$ | $\pi/2 < \cos^{-1} x \leqq \pi$ |
| $0 \leqq \tan^{-1} x < \pi/2$ | $-\pi/2 < \tan^{-1} x < 0$ |
| $0 < \cot^{-1} x \leqq \pi/2$ | $\pi/2 < \cot^{-1} x < \pi$ |
| $0 \leqq \sec^{-1} x < \pi/2$ | $\pi/2 < \sec^{-1} x \leqq \pi$ |
| $0 < \csc^{-1} x \leqq \pi/2$ | $-\pi/2 \leqq \csc^{-1} x < 0$ |

## RELATIONS BETWEEN INVERSE TRIGONOMETRIC FUNCTIONS

In all cases it is assumed that principal values are used.

**5.74** $\sin^{-1} x + \cos^{-1} x = \pi/2$

**5.75** $\tan^{-1} x + \cot^{-1} x = \pi/2$

**5.76** $\sec^{-1} x + \csc^{-1} x = \pi/2$

**5.77** $\csc^{-1} x = \sin^{-1}(1/x)$

**5.78** $\sec^{-1} x = \cos^{-1}(1/x)$

**5.79** $\cot^{-1} x = \tan^{-1}(1/x)$

**5.80** $\sin^{-1}(-x) = -\sin^{-1} x$

**5.81** $\cos^{-1}(-x) = \pi - \cos^{-1} x$

**5.82** $\tan^{-1}(-x) = -\tan^{-1} x$

**5.83** $\cot^{-1}(-x) = \pi - \cot^{-1} x$

**5.84** $\sec^{-1}(-x) = \pi - \sec^{-1} x$

**5.85** $\csc^{-1}(-x) = -\csc^{-1} x$

## GRAPHS OF INVERSE TRIGONOMETRIC FUNCTIONS

In each graph $y$ is in radians. Solid portions of curves correspond to principal values.

**5.86** $y = \sin^{-1} x$

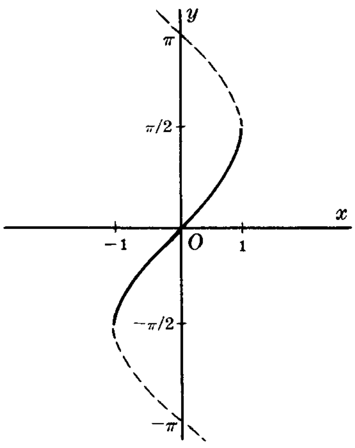


**Fig. 5-11**

**5.87** $y = \cos^{-1} x$

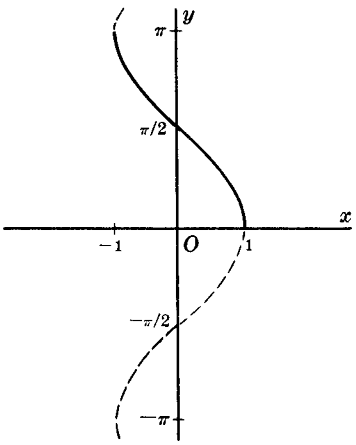


**Fig. 5-12**

**5.88** $y = \tan^{-1} x$

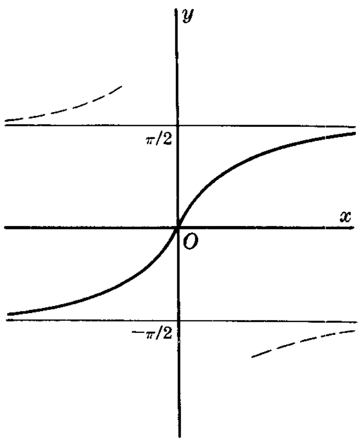


**Fig. 5-13**

**5.89** $y = \cot^{-1} x$

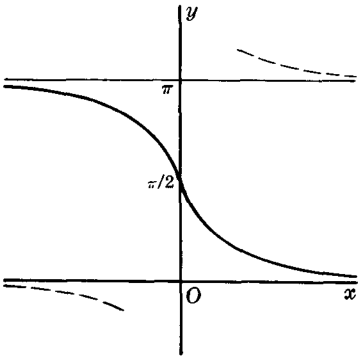


Fig. 5-14

**5.90** $y = \sec^{-1} x$

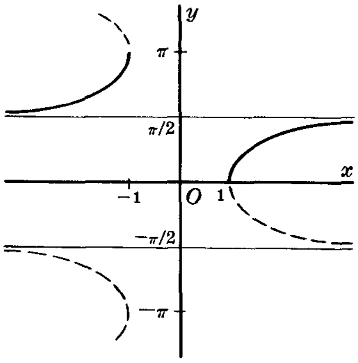


Fig. 5-15

**5.91** $y = \csc^{-1} x$

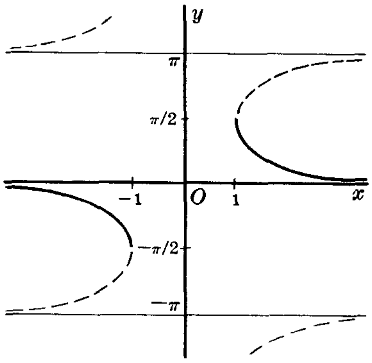


Fig. 5-16

## RELATIONSHIPS BETWEEN SIDES AND ANGLES OF A PLANE TRIANGLE

The following results hold for any plane triangle $ABC$ with sides $a, b, c$ and angles $A, B, C$.

**5.92** Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

**5.93** Law of Cosines

$$c^2 = a^2 + b^2 - 2ab \cos C$$

with similar relations involving the other sides and angles.

**5.94** Law of Tangents

$$\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}$$

with similar relations involving the other sides and angles.

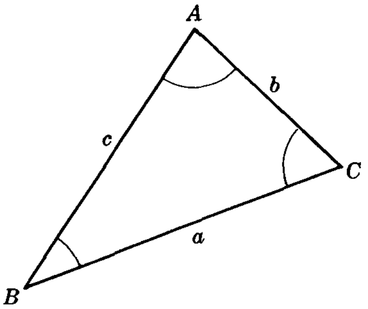


Fig. 5-17

**5.95**

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c)$ is the semiperimeter of the triangle. Similar relations involving angles $B$ and $C$ can be obtained.

See also formulas 4.5, page 5; 4.15 and 4.16, page 6.

## RELATIONSHIPS BETWEEN SIDES AND ANGLES OF A SPHERICAL TRIANGLE

Spherical triangle $ABC$ is on the surface of a sphere as shown in Fig. 5-18. Sides $a, b, c$ [which are arcs of great circles] are measured by their angles subtended at center $O$ of the sphere. $A, B, C$ are the angles opposite sides $a, b, c$ respectively. Then the following results hold.

**5.96** Law of Sines

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

**5.97** Law of Cosines

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$
$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

with similar results involving other sides and angles.

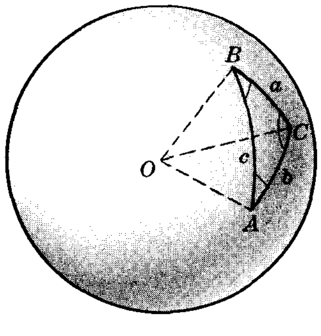


Fig. 5-18

**5.98** **Law of Tangents**

$$\frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)} = \frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}(a-b)}$$

with similar results involving other sides and angles.

**5.99**

$$\cos\frac{A}{2} = \sqrt{\frac{\sin s \sin (s-c)}{\sin b \sin c}}$$

where $s = \frac{1}{2}(a+b+c)$. Similar results hold for other sides and angles.

**5.100**

$$\cos\frac{a}{2} = \sqrt{\frac{\cos (S-B) \cos (S-C)}{\sin B \sin C}}$$

where $S = \frac{1}{2}(A+B+C)$. Similar results hold for other sides and angles.

See also formula 4.44, page 10.

## NAPIER'S RULES FOR RIGHT ANGLED SPHERICAL TRIANGLES

Except for right angle $C$, there are five parts of spherical triangle $ABC$ which if arranged in the order as given in Fig. 5-19 would be $a, b, A, c, B$.

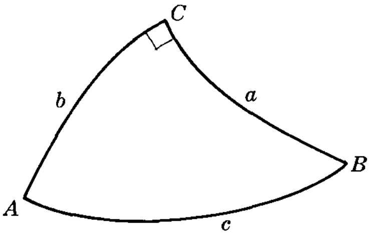


Fig. 5-19

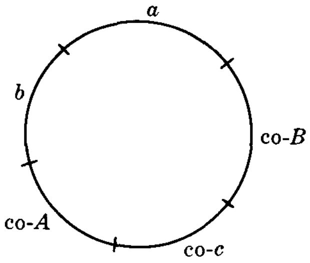


Fig. 5-20

Suppose these quantities are arranged in a circle as in Fig. 5-20 where we attach the prefix co [indicating *complement*] to hypotenuse $c$ and angles $A$ and $B$.

Any one of the parts of this circle is called a *middle part*, the two neighboring parts are called *adjacent parts* and the two remaining parts are called *opposite parts*. Then Napier's rules are

**5.101** The sine of any middle part equals the product of the tangents of the adjacent parts.

**5.102** The sine of any middle part equals the product of the cosines of the opposite parts.

**Example:** Since co-$A = 90° - A$, co-$B = 90° - B$, we have

$$\sin a = \tan b \tan (\text{co-}B) \quad \text{or} \quad \sin a = \tan b \cot B$$

$$\sin (\text{co-}A) = \cos a \cos (\text{co-}B) \quad \text{or} \quad \cos A = \cos a \sin B$$

These can of course be obtained also from the results 5.97 on page 19.

# 6 COMPLEX NUMBERS

## DEFINITIONS INVOLVING COMPLEX NUMBERS

A *complex number* is generally written as $a + bi$ where $a$ and $b$ are real numbers and $i$, called the *imaginary unit*, has the property that $i^2 = -1$. The real numbers $a$ and $b$ are called the *real* and *imaginary parts* of $a + bi$ respectively.

The complex numbers $a + bi$ and $a - bi$ are called *complex conjugates* of each other.

## EQUALITY OF COMPLEX NUMBERS

**6.1** $$a + bi = c + di \quad \text{if and only if} \quad a = c \text{ and } b = d$$

## ADDITION OF COMPLEX NUMBERS

**6.2** $$(a + bi) + (c + di) = (a + c) + (b + d)i$$

## SUBTRACTION OF COMPLEX NUMBERS

**6.3** $$(a + bi) - (c + di) = (a - c) + (b - d)i$$

## MULTIPLICATION OF COMPLEX NUMBERS

**6.4** $$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

## DIVISION OF COMPLEX NUMBERS

**6.5** $$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac + bd}{c^2 + d^2} + \left(\frac{bc - ad}{c^2 + d^2}\right) i$$

Note that the above operations are obtained by using the ordinary rules of algebra and replacing $i^2$ by $-1$ wherever it occurs.

## GRAPH OF A COMPLEX NUMBER

A complex number $a + bi$ can be plotted as a point $(a, b)$ on an $xy$ plane called an *Argand diagram* or *Gaussian plane.* For example in Fig. 6-1 $P$ represents the complex number $-3 + 4i$.

A complex number can also be interpreted as a *vector* from $O$ to $P$.

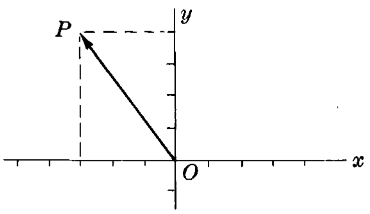


**Fig. 6-1**

## POLAR FORM OF A COMPLEX NUMBER

In Fig. 6-2 point $P$ with coordinates $(x, y)$ represents the complex number $x + iy$. Point $P$ can also be represented by *polar coordinates* $(r, \theta)$. Since $x = r\cos\theta,\ y = r\sin\theta$ we have

**6.6**
$$x + iy = r(\cos\theta + i\sin\theta)$$

called the *polar form* of the complex number. We often call $r = \sqrt{x^2 + y^2}$ the *modulus* and $\theta$ the *amplitude* of $x + iy$.

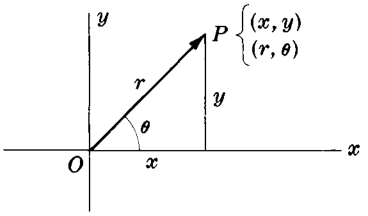


**Fig. 6-2**

## MULTIPLICATION AND DIVISION OF COMPLEX NUMBERS IN POLAR FORM

**6.7**
$$[r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)] = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

**6.8**
$$\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

## DE MOIVRE'S THEOREM

If $p$ is any real number, De Moivre's theorem states that

**6.9**
$$[r(\cos\theta + i\sin\theta)]^p = r^p(\cos p\theta + i\sin p\theta)$$

## ROOTS OF COMPLEX NUMBERS

If $p = 1/n$ where $n$ is any positive integer, 6.9 can be written

**6.10**
$$[r(\cos\theta + i\sin\theta)]^{1/n} = r^{1/n}\left[\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right]$$

where $k$ is any integer. From this the $n$ $n$th roots of a complex number can be obtained by putting $k = 0, 1, 2, \ldots, n-1$.

# 7 EXPONENTIAL and LOGARITHMIC FUNCTIONS

## LAWS OF EXPONENTS

In the following $p, q$ are real numbers, $a, b$ are positive numbers and $m, n$ are positive integers.

**7.1** $a^p \cdot a^q = a^{p+q}$ **7.2** $a^p/a^q = a^{p-q}$ **7.3** $(a^p)^q = a^{pq}$

**7.4** $a^0 = 1, \; a \neq 0$ **7.5** $a^{-p} = 1/a^p$ **7.6** $(ab)^p = a^p b^p$

**7.7** $\sqrt[n]{a} = a^{1/n}$ **7.8** $\sqrt[n]{a^m} = a^{m/n}$ **7.9** $\sqrt[n]{a/b} = \sqrt[n]{a}/\sqrt[n]{b}$

In $a^p$, $p$ is called the *exponent*, $a$ is the *base* and $a^p$ is called the $p$th *power of* $a$. The function $y = a^x$ is called an *exponential function.*

## LOGARITHMS AND ANTILOGARITHMS

If $a^p = N$ where $a \neq 0$ or 1, then $p = \log_a N$ is called the *logarithm* of $N$ to the base $a$. The number $N = a^p$ is called the *antilogarithm* of $p$ to the base $a$, written $\operatorname{antilog}_a p$.

**Example:** Since $3^2 = 9$ we have $\log_3 9 = 2$, $\operatorname{antilog}_3 2 = 9$.

The function $y = \log_a x$ is called a *logarithmic function.*

## LAWS OF LOGARITHMS

**7.10** $$\log_a MN = \log_a M + \log_a N$$

**7.11** $$\log_a \frac{M}{N} = \log_a M - \log_a N$$

**7.12** $$\log_a M^p = p \log_a M$$

## COMMON LOGARITHMS AND ANTILOGARITHMS

**Common logarithms and antilogarithms [also called *Briggsian*] are those in which the base $a = 10$. The common logarithm of $N$ is denoted by $\log_{10} N$ or briefly $\log N$. For tables of common logarithms and antilogarithms, see pages 202-205. For illustrations using these tables see pages 194-196.**

## NATURAL LOGARITHMS AND ANTILOGARITHMS

Natural logarithms and antilogarithms [also called Napierian] are those in which the base $a = e = 2.71828\,18\ldots$ [see page 1]. The natural logarithm of $N$ is denoted by $\log_e N$ or $\ln N$. For tables of natural logarithms see pages 224-225. For tables of natural antilogarithms [i.e. tables giving $e^x$ for values of $x$] see pages 226-227. For illustrations using these tables see pages 196 and 200.

## CHANGE OF BASE OF LOGARITHMS

The relationship between logarithms of a number $N$ to different bases $a$ and $b$ is given by

**7.13** $$\log_a N = \frac{\log_b N}{\log_b a}$$

In particular,

**7.14** $$\log_e N = \ln N = 2.30258\,50929\,94\ldots \log_{10} N$$

**7.15** $$\log_{10} N = \log N = 0.43429\,44819\,03\ldots \log_e N$$

## RELATIONSHIP BETWEEN EXPONENTIAL AND TRIGONOMETRIC FUNCTIONS

**7.16** $$e^{i\theta} = \cos\theta + i\sin\theta, \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

These are called *Euler's identities*. Here $i$ is the imaginary unit [see page 21].

**7.17** $$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

**7.18** $$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

**7.19** $$\tan\theta = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})} = -i\left(\frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}\right)$$

**7.20** $$\cot\theta = i\left(\frac{e^{i\theta} + e^{-i\theta}}{e^{i\theta} - e^{-i\theta}}\right)$$

**7.21** $$\sec\theta = \frac{2}{e^{i\theta} + e^{-i\theta}}$$

**7.22** $$\csc\theta = \frac{2i}{e^{i\theta} - e^{-i\theta}}$$

## PERIODICITY OF EXPONENTIAL FUNCTIONS

**7.23** $$e^{i(\theta + 2k\pi)} = e^{i\theta} \qquad k = \text{integer}$$

From this it is seen that $e^x$ has period $2\pi i$.

## POLAR FORM OF COMPLEX NUMBERS EXPRESSED AS AN EXPONENTIAL

The polar form of a complex number $x + iy$ can be written in terms of exponentials [see 6.6, page 22] as

**7.24** $$x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

## OPERATIONS WITH COMPLEX NUMBERS IN POLAR FORM

Formulas 6.7 through 6.10 on page 22 are equivalent to the following.

**7.25** $$(r_1e^{i\theta_1})(r_2e^{i\theta_2}) = r_1r_2e^{i(\theta_1+\theta_2)}$$

**7.26** $$\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$$

**7.27** $$(re^{i\theta})^p = r^pe^{ip\theta} \qquad \text{[De Moivre's theorem]}$$

**7.28** $$(re^{i\theta})^{1/n} = [re^{i(\theta+2k\pi)}]^{1/n} = r^{1/n}e^{i(\theta+2k\pi)/n}$$

## LOGARITHM OF A COMPLEX NUMBER

**7.29** $$\ln(re^{i\theta}) = \ln r + i\theta + 2k\pi i \qquad k = \text{integer}$$

# 8

# HYPERBOLIC FUNCTIONS

## DEFINITION OF HYPERBOLIC FUNCTIONS

**8.1** *Hyperbolic sine* of $x$ $= \sinh x = \dfrac{e^x - e^{-x}}{2}$

**8.2** *Hyperbolic cosine* of $x$ $= \cosh x = \dfrac{e^x + e^{-x}}{2}$

**8.3** *Hyperbolic tangent* of $x$ $= \tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

**8.4** *Hyperbolic cotangent* of $x$ $= \coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

**8.5** *Hyperbolic secant* of $x$ $= \operatorname{sech} x = \dfrac{2}{e^x + e^{-x}}$

**8.6** *Hyperbolic cosecant* of $x$ $= \operatorname{csch} x = \dfrac{2}{e^x - e^{-x}}$

## RELATIONSHIPS AMONG HYPERBOLIC FUNCTIONS

**8.7** $\tanh x = \dfrac{\sinh x}{\cosh x}$

**8.8** $\coth x = \dfrac{1}{\tanh x} = \dfrac{\cosh x}{\sinh x}$

**8.9** $\operatorname{sech} x = \dfrac{1}{\cosh x}$

**8.10** $\operatorname{csch} x = \dfrac{1}{\sinh x}$

**8.11** $\cosh^2 x - \sinh^2 x = 1$

**8.12** $\operatorname{sech}^2 x + \tanh^2 x = 1$

**8.13** $\coth^2 x - \operatorname{csch}^2 x = 1$

## FUNCTIONS OF NEGATIVE ARGUMENTS

**8.14** $\sinh(-x) = -\sinh x$

**8.15** $\cosh(-x) = \cosh x$

**8.16** $\tanh(-x) = -\tanh x$

**8.17** $\operatorname{csch}(-x) = -\operatorname{csch} x$

**8.18** $\operatorname{sech}(-x) = \operatorname{sech} x$

**8.19** $\coth(-x) = -\coth x$

## ADDITION FORMULAS

**8.20** $$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

**8.21** $$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

**8.22** $$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

**8.23** $$\coth(x \pm y) = \frac{\coth x \coth y \pm 1}{\coth y \pm \coth x}$$

## DOUBLE ANGLE FORMULAS

**8.24** $$\sinh 2x = 2 \sinh x \cosh x$$

**8.25** $$\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$$

**8.26** $$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$$

## HALF ANGLE FORMULAS

**8.27** $$\sinh \frac{x}{2} = \pm\sqrt{\frac{\cosh x - 1}{2}} \qquad [+ \text{ if } x > 0, \; - \text{ if } x < 0]$$

**8.28** $$\cosh \frac{x}{2} = \sqrt{\frac{\cosh x + 1}{2}}$$

**8.29** $$\tanh \frac{x}{2} = \pm\sqrt{\frac{\cosh x - 1}{\cosh x + 1}} \qquad [+ \text{ if } x > 0, \; - \text{ if } x < 0]$$
$$= \frac{\sinh x}{\cosh x + 1} = \frac{\cosh x - 1}{\sinh x}$$

## MULTIPLE ANGLE FORMULAS

**8.30** $$\sinh 3x = 3 \sinh x + 4 \sinh^3 x$$

**8.31** $$\cosh 3x = 4\cosh^3 x - 3 \cosh x$$

**8.32** $$\tanh 3x = \frac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x}$$

**8.33** $$\sinh 4x = 8 \sinh^3 x \cosh x + 4 \sinh x \cosh x$$

**8.34** $$\cosh 4x = 8 \cosh^4 x - 8 \cosh^2 x + 1$$

**8.35** $$\tanh 4x = \frac{4 \tanh x + 4 \tanh^3 x}{1 + 6 \tanh^2 x + \tanh^4 x}$$

## POWERS OF HYPERBOLIC FUNCTIONS

**8.36** $\sinh^2 x = \tfrac{1}{2}\cosh 2x - \tfrac{1}{2}$

**8.37** $\cosh^2 x = \tfrac{1}{2}\cosh 2x + \tfrac{1}{2}$

**8.38** $\sinh^3 x = \tfrac{1}{4}\sinh 3x - \tfrac{3}{4}\sinh x$

**8.39** $\cosh^3 x = \tfrac{1}{4}\cosh 3x + \tfrac{3}{4}\cosh x$

**8.40** $\sinh^4 x = \tfrac{3}{8} - \tfrac{1}{2}\cosh 2x + \tfrac{1}{8}\cosh 4x$

**8.41** $\cosh^4 x = \tfrac{3}{8} + \tfrac{1}{2}\cosh 2x + \tfrac{1}{8}\cosh 4x$

## SUM, DIFFERENCE AND PRODUCT OF HYPERBOLIC FUNCTIONS

**8.42** $\sinh x + \sinh y = 2\sinh\tfrac{1}{2}(x+y)\cosh\tfrac{1}{2}(x-y)$

**8.43** $\sinh x - \sinh y = 2\cosh\tfrac{1}{2}(x+y)\sinh\tfrac{1}{2}(x-y)$

**8.44** $\cosh x + \cosh y = 2\cosh\tfrac{1}{2}(x+y)\cosh\tfrac{1}{2}(x-y)$

**8.45** $\cosh x - \cosh y = 2\sinh\tfrac{1}{2}(x+y)\sinh\tfrac{1}{2}(x-y)$

**8.46** $\sinh x\sinh y = \tfrac{1}{2}\{\cosh(x+y) - \cosh(x-y)\}$

**8.47** $\cosh x\cosh y = \tfrac{1}{2}\{\cosh(x+y) + \cosh(x-y)\}$

**8.48** $\sinh x\cosh y = \tfrac{1}{2}\{\sinh(x+y) + \sinh(x-y)\}$

## EXPRESSION OF HYPERBOLIC FUNCTIONS IN TERMS OF OTHERS

In the following we assume $x > 0$. If $x < 0$ use the appropriate sign as indicated by formulas 8.14 to 8.19.

| | $\sinh x = u$ | $\cosh x = u$ | $\tanh x = u$ | $\coth x = u$ | $\operatorname{sech} x = u$ | $\operatorname{csch} x = u$ |
|---|---|---|---|---|---|---|
| $\sinh x$ | $u$ | $\sqrt{u^2-1}$ | $u/\sqrt{1-u^2}$ | $1/\sqrt{u^2-1}$ | $\sqrt{1-u^2}/u$ | $1/u$ |
| $\cosh x$ | $\sqrt{1+u^2}$ | $u$ | $1/\sqrt{1-u^2}$ | $u/\sqrt{u^2-1}$ | $1/u$ | $\sqrt{1+u^2}/u$ |
| $\tanh x$ | $u/\sqrt{1+u^2}$ | $\sqrt{u^2-1}/u$ | $u$ | $1/u$ | $\sqrt{1-u^2}$ | $1/\sqrt{1+u^2}$ |
| $\coth x$ | $\sqrt{u^2+1}/u$ | $u/\sqrt{u^2-1}$ | $1/u$ | $u$ | $1/\sqrt{1-u^2}$ | $\sqrt{1+u^2}$ |
| $\operatorname{sech} x$ | $1/\sqrt{1+u^2}$ | $1/u$ | $\sqrt{1-u^2}$ | $\sqrt{u^2-1}/u$ | $u$ | $u/\sqrt{1+u^2}$ |
| $\operatorname{csch} x$ | $1/u$ | $1/\sqrt{u^2-1}$ | $\sqrt{1-u^2}/u$ | $\sqrt{u^2-1}$ | $u/\sqrt{1-u^2}$ | $u$ |

## GRAPHS OF HYPERBOLIC FUNCTIONS

**8.49** $y = \sinh x$

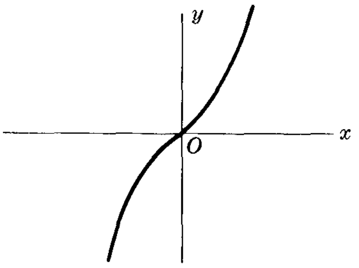


Fig. 8-1

**8.50** $y = \cosh x$

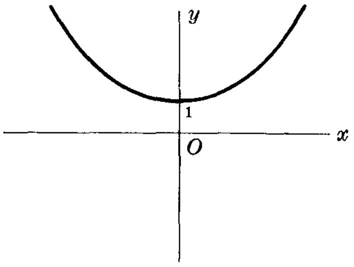


Fig. 8-2

**8.51** $y = \tanh x$

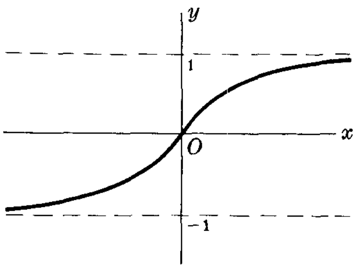


Fig. 8-3

**8.52** $y = \coth x$

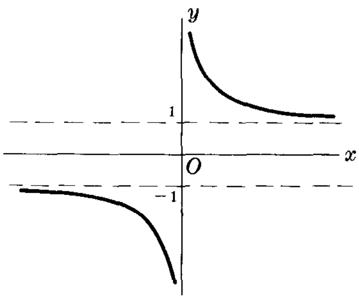


Fig. 8-4

**8.53** $y = \operatorname{sech} x$

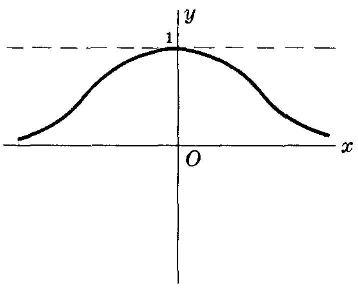


Fig. 8-5

**8.54** $y = \operatorname{csch} x$

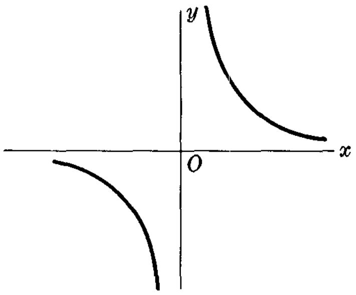


Fig. 8-6

## INVERSE HYPERBOLIC FUNCTIONS

If $x = \sinh y$, then $y = \sinh^{-1} x$ is called the *inverse hyperbolic sine* of $x$. Similarly we define the other inverse hyperbolic functions. The inverse hyperbolic functions are multiple-valued and as in the case of inverse trigonometric functions [see page 17] we restrict ourselves to principal values for which they can be considered as single-valued.

The following list shows the principal values [unless otherwise indicated] of the inverse hyperbolic functions expressed in terms of logarithmic functions which are taken as real valued.

**8.55** $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$ $\quad -\infty < x < \infty$

**8.56** $\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$ $\quad x \geqq 1$ $\quad [\cosh^{-1} x > 0$ is principal value$]$

**8.57** $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ $\quad -1 < x < 1$

**8.58** $\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$ $\quad x > 1$ or $x < -1$

**8.59** $\operatorname{sech}^{-1} x = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right)$ $\quad 0 < x \leqq 1$ $\quad [\operatorname{sech}^{-1} x > 0$ is principal value$]$

**8.60** $\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right)$ $\quad x \neq 0$

## RELATIONS BETWEEN INVERSE HYPERBOLIC FUNCTIONS

**8.61** $\operatorname{csch}^{-1} x = \sinh^{-1}(1/x)$

**8.62** $\operatorname{sech}^{-1} x = \cosh^{-1}(1/x)$

**8.63** $\coth^{-1} x = \tanh^{-1}(1/x)$

**8.64** $\sinh^{-1}(-x) = -\sinh^{-1} x$

**8.65** $\tanh^{-1}(-x) = -\tanh^{-1} x$

**8.66** $\coth^{-1}(-x) = -\coth^{-1} x$

**8.67** $\operatorname{csch}^{-1}(-x) = -\operatorname{csch}^{-1} x$

## GRAPHS OF INVERSE HYPERBOLIC FUNCTIONS

**8.68** $y = \sinh^{-1} x$

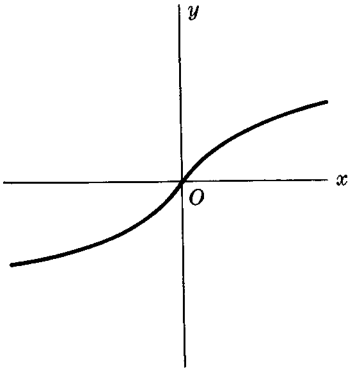


Fig. 8-7

**8.69** $y = \cosh^{-1} x$

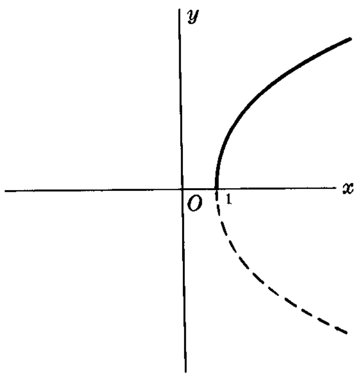


Fig. 8-8

**8.70** $y = \tanh^{-1} x$

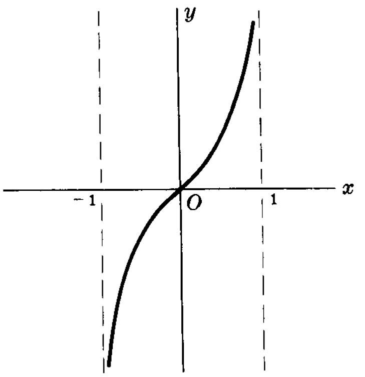


Fig. 8-9

**8.71** $y = \coth^{-1} x$

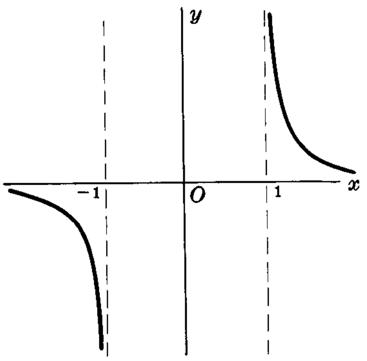


Fig. 8-10

**8.72** $y = \operatorname{sech}^{-1} x$

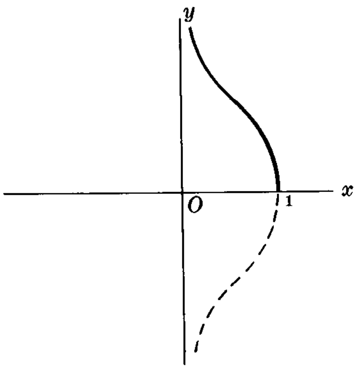


Fig. 8-11

**8.73** $y = \operatorname{csch}^{-1} x$

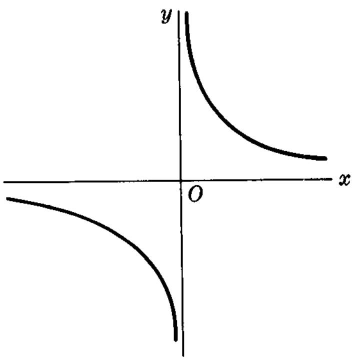


Fig. 8-12

## RELATIONSHIP BETWEEN HYPERBOLIC AND TRIGONOMETRIC FUNCTIONS

**8.74** $\sin(ix) = i\sinh x$

**8.75** $\cos(ix) = \cosh x$

**8.76** $\tan(ix) = i\tanh x$

**8.77** $\csc(ix) = -i\,\mathrm{csch}\,x$

**8.78** $\sec(ix) = \mathrm{sech}\,x$

**8.79** $\cot(ix) = -i\coth x$

**8.80** $\sinh(ix) = i\sin x$

**8.81** $\cosh(ix) = \cos x$

**8.82** $\tanh(ix) = i\tan x$

**8.83** $\mathrm{csch}(ix) = -i\csc x$

**8.84** $\mathrm{sech}(ix) = \sec x$

**8.85** $\coth(ix) = -i\cot x$

## PERIODICITY OF HYPERBOLIC FUNCTIONS

In the following $k$ is any integer.

**8.86** $\sinh(x + 2k\pi i) = \sinh x$

**8.87** $\cosh(x + 2k\pi i) = \cosh x$

**8.88** $\tanh(x + k\pi i) = \tanh x$

**8.89** $\mathrm{csch}(x + 2k\pi i) = \mathrm{csch}\,x$

**8.90** $\mathrm{sech}(x + 2k\pi i) = \mathrm{sech}\,x$

**8.91** $\coth(x + k\pi i) = \coth x$

## RELATIONSHIP BETWEEN INVERSE HYPERBOLIC AND INVERSE TRIGONOMETRIC FUNCTIONS

**8.92** $\sin^{-1}(ix) = i\sinh^{-1} x$

**8.93** $\sinh^{-1}(ix) = i\sin^{-1} x$

**8.94** $\cos^{-1} x = \pm i\cosh^{-1} x$

**8.95** $\cosh^{-1} x = \pm i\cos^{-1} x$

**8.96** $\tan^{-1}(ix) = i\tanh^{-1} x$

**8.97** $\tanh^{-1}(ix) = i\tan^{-1} x$

**8.98** $\cot^{-1}(ix) = -i\coth^{-1} x$

**8.99** $\coth^{-1}(ix) = -i\cot^{-1} x$

**8.100** $\sec^{-1} x = \pm i\,\mathrm{sech}^{-1} x$

**8.101** $\mathrm{sech}^{-1} x = \pm i\sec^{-1} x$

**8.102** $\csc^{-1}(ix) = -i\,\mathrm{csch}^{-1} x$

**8.103** $\mathrm{csch}^{-1}(ix) = -i\csc^{-1} x$

# 9 SOLUTIONS of ALGEBRAIC EQUATIONS

## QUADRATIC EQUATION: $ax^2 + bx + c = 0$

**9.1** Solutions:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $a, b, c$ are real and if $D = b^2 - 4ac$ is the *discriminant*, then the roots are

(i) real and unequal if $D > 0$

(ii) real and equal if $D = 0$

(iii) complex conjugate if $D < 0$

**9.2** If $x_1, x_2$ are the roots, then $x_1 + x_2 = -b/a$ and $x_1x_2 = c/a$.

## CUBIC EQUATION: $x^3 + a_1x^2 + a_2x + a_3 = 0$

Let

$$Q = \frac{3a_2 - a_1^2}{9}, \quad R = \frac{9a_1a_2 - 27a_3 - 2a_1^3}{54},$$

$$S = \sqrt[3]{R + \sqrt{Q^3 + R^2}}, \quad T = \sqrt[3]{R - \sqrt{Q^3 + R^2}}$$

**9.3** Solutions:

$$\begin{cases} x_1 = S + T - \tfrac{1}{3}a_1 \\ x_2 = -\tfrac{1}{2}(S + T) - \tfrac{1}{3}a_1 + \tfrac{1}{2}i\sqrt{3}\,(S - T) \\ x_3 = -\tfrac{1}{2}(S + T) - \tfrac{1}{3}a_1 - \tfrac{1}{2}i\sqrt{3}\,(S - T) \end{cases}$$

If $a_1, a_2, a_3$ are real and if $D = Q^3 + R^2$ is the *discriminant*, then

(i) one root is real and two complex conjugate if $D > 0$

(ii) all roots are real and at least two are equal if $D = 0$

(iii) all roots are real and unequal if $D < 0$.

If $D < 0$, computation is simplified by use of trigonometry.

**9.4** Solutions if $D < 0$:

$$\begin{cases} x_1 = 2\sqrt{-Q}\cos(\tfrac{1}{3}\theta) \\ x_2 = 2\sqrt{-Q}\cos(\tfrac{1}{3}\theta + 120^\circ) \\ x_3 = 2\sqrt{-Q}\cos(\tfrac{1}{3}\theta + 240^\circ) \end{cases} \qquad \text{where } \cos\theta = -R/\sqrt{-Q^3}$$

**9.5**

$$x_1 + x_2 + x_3 = -a_1, \quad x_1x_2 + x_2x_3 + x_3x_1 = a_2, \quad x_1x_2x_3 = -a_3$$

where $x_1, x_2, x_3$ are the three roots.

## QUARTIC EQUATION: $x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$

Let $y_1$ be a real root of the cubic equation

**9.6** $$y^3 - a_2y^2 + (a_1a_3 - 4a_4)y + (4a_2a_4 - a_3^2 - a_1^2a_4) = 0$$

**9.7** **Solutions:** The 4 roots of $z^2 + \frac{1}{2}\{a_1 \pm \sqrt{a_1^2 - 4a_2 + 4y_1}\}z + \frac{1}{2}\{y_1 \pm \sqrt{y_1^2 - 4a_4}\} = 0$

If all roots of 9.6 are real, computation is simplified by using that particular real root which produces all real coefficients in the quadratic equation 9.7.

**9.8** $$\begin{cases} x_1 + x_2 + x_3 + x_4 = -a_1 \\ x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 + x_1x_3 + x_2x_4 = a_2 \\ x_1x_2x_3 + x_2x_3x_4 + x_1x_2x_4 + x_1x_3x_4 = -a_3 \\ x_1x_2x_3x_4 = a_4 \end{cases}$$

$$\frac{Q - \frac{1}{9}}{R}$$

where $x_1, x_2, x_3, x_4$ are the four roots.

$a_2 = 0$

$a_1$ $a_3$

$x^3 - x^2 - 1 = 0$ $x^2 - x^3 = -1$

$x^3$ $a_1x^2$ $a_3$ $16+9 = 25$

$0 - 27 -$

$\cancel{Q_1 Q_2 Q_3}$

$x^3 - x^2 - 1 = 0$

$Q = \frac{0-1}{9}$ $-\frac{1}{9}$

$\cancel{0-1}$ $= \frac{1}{9}$

$\frac{1+\sqrt{5}}{2}$ $x^3 - x^2 = 1$

$\frac{x+1}{x-1}$ $\frac{1+\sqrt[3]{5}}{2}$

$x^2 + x - x - 1$ $x^3 = x^2 + 1$

$x^2 - 1$ $x^3 = \sqrt{x^3}$ $x = \sqrt[3]{x^2+1}$

# 10 FORMULAS from PLANE ANALYTIC GEOMETRY

## DISTANCE $d$ BETWEEN TWO POINTS $P_1(x_1, y_1)$ AND $P_2(x_2, y_2)$

**10.1** $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

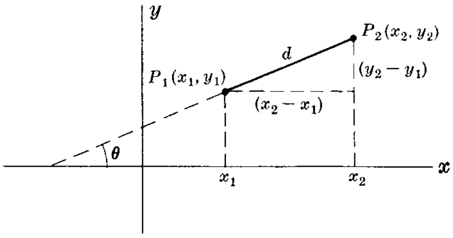


Fig. 10-1

## SLOPE $m$ OF LINE JOINING TWO POINTS $P_1(x_1, y_1)$ AND $P_2(x_2, y_2)$

**10.2** $$m = \frac{y_2 - y_1}{x_2 - x_1} = \tan\theta$$

## EQUATION OF LINE JOINING TWO POINTS $P_1(x_1, y_1)$ AND $P_2(x_2, y_2)$

**10.3** $$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} = m \quad \text{or} \quad y - y_1 = m(x - x_1)$$

**10.4** $$y = mx + b$$

where $b = y_1 - mx_1 = \dfrac{x_2y_1 - x_1y_2}{x_2 - x_1}$ is the *intercept* on the $y$ axis, i.e. the $y$ *intercept.*

## EQUATION OF LINE IN TERMS OF $x$ INTERCEPT $a \neq 0$ AND $y$ INTERCEPT $b \neq 0$

**10.5** $$\frac{x}{a} + \frac{y}{b} = 1$$

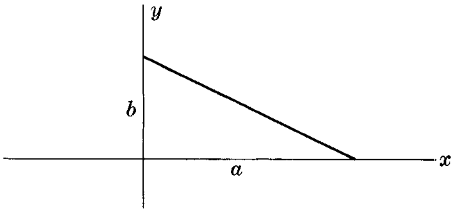


Fig. 10-2

## NORMAL FORM FOR EQUATION OF LINE

**10.6** $$x\cos\alpha + y\sin\alpha = p$$

where $p$ = perpendicular distance from origin $O$ to line

and $\alpha$ = angle of inclination of perpendicular with positive $x$ axis.

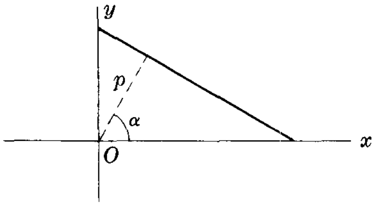


Fig. 10-3

## GENERAL EQUATION OF LINE

**10.7** $$Ax + By + C = 0$$

## DISTANCE FROM POINT $(x_1, y_1)$ TO LINE $Ax + By + C = 0$

**10.8** $$\frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}$$

where the sign is chosen so that the distance is nonnegative.

## ANGLE $\psi$ BETWEEN TWO LINES HAVING SLOPES $m_1$ AND $m_2$

**10.9** $$\tan\psi = \frac{m_2 - m_1}{1 + m_1 m_2}$$

Lines are parallel or coincident if and only if $m_1 = m_2$.

Lines are perpendicular if and only if $m_2 = -1/m_1$.

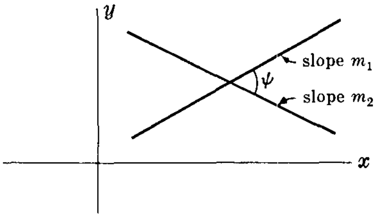


Fig. 10-4

## AREA OF TRIANGLE WITH VERTICES AT $(x_1, y_1), (x_2, y_2), (x_3, y_3)$

**10.10** $$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \pm\frac{1}{2}(x_1y_2 + y_1x_3 + y_3x_2 - y_2x_3 - y_1x_2 - x_1y_3)$$

where the sign is chosen so that the area is nonnegative.

If the area is zero the points all lie on a line.

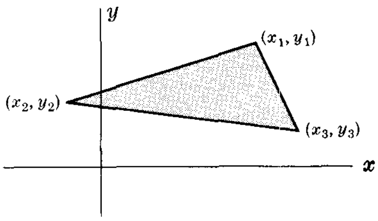


Fig. 10-5

## TRANSFORMATION OF COORDINATES INVOLVING PURE TRANSLATION

**10.11**

$$\begin{cases} x = x' + x_0 \\ y = y' + y_0 \end{cases} \quad \text{or} \quad \begin{cases} x' = x - x_0 \\ y' = y - y_0 \end{cases}$$

where $(x, y)$ are old coordinates [i.e. coordinates relative to $xy$ system], $(x', y')$ are new coordinates [relative to $x'y'$ system] and $(x_0, y_0)$ are the coordinates of the new origin $O'$ relative to the old $xy$ coordinate system.

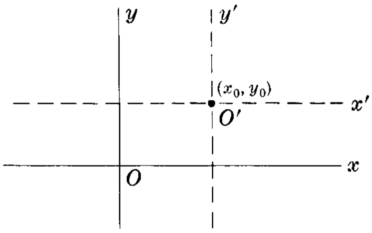


Fig. 10-6

## TRANSFORMATION OF COORDINATES INVOLVING PURE ROTATION

**10.12**

$$\begin{cases} x = x' \cos\alpha - y' \sin\alpha \\ y = x' \sin\alpha + y' \cos\alpha \end{cases} \quad \text{or} \quad \begin{cases} x' = x\cos\alpha + y\sin\alpha \\ y' = y\cos\alpha - x\sin\alpha \end{cases}$$

where the origins of the old $[xy]$ and new $[x'y']$ coordinate systems are the same but the $x'$ axis makes an angle $\alpha$ with the positive $x$ axis.

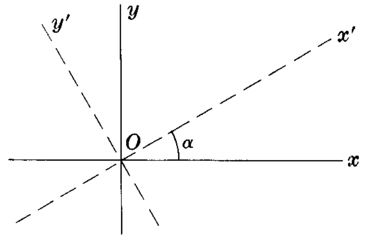


Fig. 10-7

## TRANSFORMATION OF COORDINATES INVOLVING TRANSLATION AND ROTATION

**10.13**

$$\begin{cases} x = x' \cos\alpha - y' \sin\alpha + x_0 \\ y = x' \sin\alpha + y' \cos\alpha + y_0 \end{cases}$$

or

$$\begin{cases} x' = (x - x_0)\cos\alpha + (y - y_0)\sin\alpha \\ y' = (y - y_0)\cos\alpha - (x - x_0)\sin\alpha \end{cases}$$

where the new origin $O'$ of $x'y'$ coordinate system has coordinates $(x_0, y_0)$ relative to the old $xy$ coordinate system and the $x'$ axis makes an angle $\alpha$ with the positive $x$ axis.

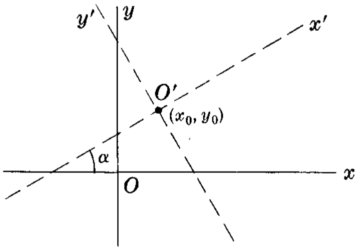


Fig. 10-8

## POLAR COORDINATES $(r, \theta)$

A point $P$ can be located by rectangular coordinates $(x, y)$ or polar coordinates $(r, \theta)$. The transformation between these coordinates is

**10.14**

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \text{or} \quad \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}(y/x) \end{cases}$$

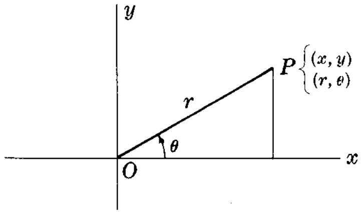


Fig. 10-9

## EQUATION OF CIRCLE OF RADIUS $R$, CENTER AT $(x_0, y_0)$

**10.15** $$(x-x_0)^2 + (y-y_0)^2 = R^2$$

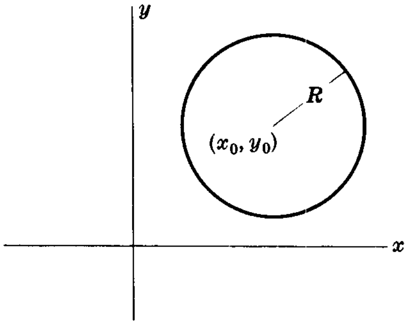


Fig. 10-10

## EQUATION OF CIRCLE OF RADIUS $R$ PASSING THROUGH ORIGIN

**10.16** $$r = 2R\cos(\theta - \alpha)$$

where $(r, \theta)$ are polar coordinates of any point on the circle and $(R, \alpha)$ are polar coordinates of the center of the circle.

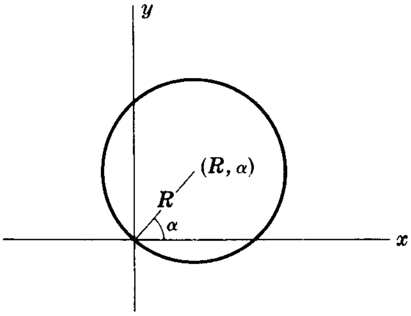


Fig. 10-11

## CONICS [ELLIPSE, PARABOLA OR HYPERBOLA]

If a point $P$ moves so that its distance from a fixed point [called the *focus*] divided by its distance from a fixed line [called the *directrix*] is a constant $\epsilon$ [called the *eccentricity*], then the curve described by $P$ is called a *conic* [so-called because such curves can be obtained by intersecting a plane and a cone at different angles].

If the focus is chosen at origin $O$ the equation of a conic in polar coordinates $(r, \theta)$ is, if $OQ = p$ and $LM = D$, [see Fig. 10-12]

**10.17** $$r = \frac{p}{1-\epsilon\cos\theta} = \frac{\epsilon D}{1-\epsilon\cos\theta}$$

The conic is

(i) an ellipse if $\epsilon < 1$

(ii) a parabola if $\epsilon = 1$

(iii) a hyperbola if $\epsilon > 1$.

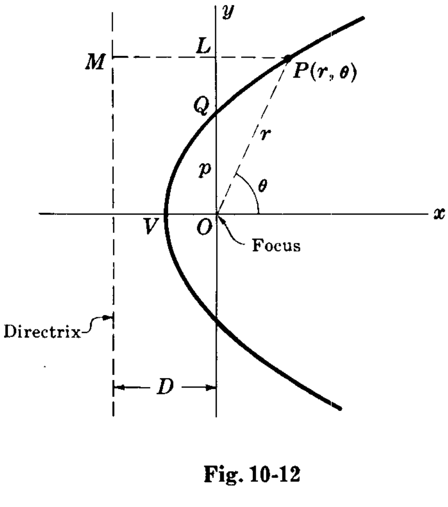


Fig. 10-12

## ELLIPSE WITH CENTER $C(x_0, y_0)$ AND MAJOR AXIS PARALLEL TO $x$ AXIS

**10.18** Length of major axis $A'A = 2a$

**10.19** Length of minor axis $B'B = 2b$

**10.20** Distance from center $C$ to focus $F$ or $F'$ is

$$c = \sqrt{a^2 - b^2}$$

**10.21** Eccentricity $= \epsilon = \dfrac{c}{a} = \dfrac{\sqrt{a^2 - b^2}}{a}$

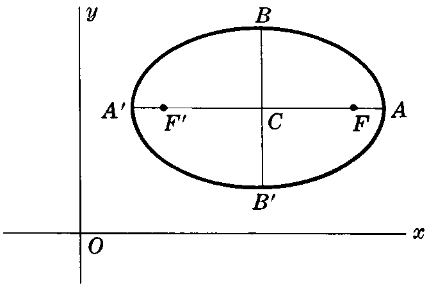


Fig. 10-13

**10.22** Equation in rectangular coordinates:

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1$$

**10.23** Equation in polar coordinates if $C$ is at $O$: $\quad r^2 = \dfrac{a^2b^2}{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$

**10.24** Equation in polar coordinates if $C$ is on $x$ axis and $F'$ is at $O$: $\quad r = \dfrac{a(1 - \epsilon^2)}{1 - \epsilon \cos \theta}$

**10.25** If $P$ is any point on the ellipse, $PF + PF' = 2a$

If the major axis is parallel to the $y$ axis, interchange $x$ and $y$ in the above or replace $\theta$ by $\frac{1}{2}\pi - \theta$ [or $90° - \theta$].

## PARABOLA WITH AXIS PARALLEL TO $x$ AXIS

If vertex is at $A(x_0, y_0)$ and the distance from $A$ to focus $F$ is $a > 0$, the equation of the parabola is

**10.26** $\quad (y - y_0)^2 = 4a(x - x_0) \quad$ if parabola opens to right [Fig. 10-14]

**10.27** $\quad (y - y_0)^2 = -4a(x - x_0) \quad$ if parabola opens to left [Fig. 10-15]

If focus is at the origin [Fig. 10-16] the equation in polar coordinates is

**10.28**
$$r = \frac{2a}{1 - \cos \theta}$$

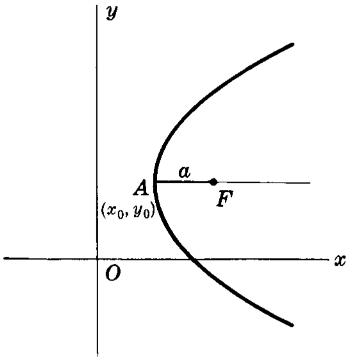


Fig. 10-14

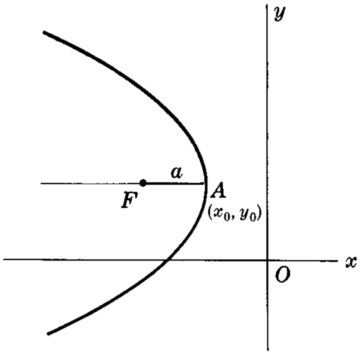


Fig. 10-15

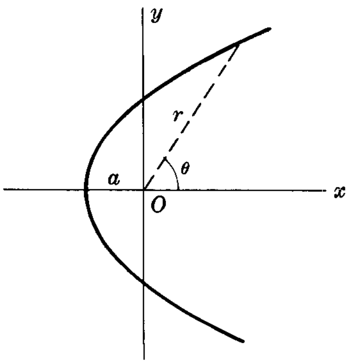


Fig. 10-16

In case the axis is parallel to the $y$ axis, interchange $x$ and $y$ or replace $\theta$ by $\frac{1}{2}\pi - \theta$ [or $90° - \theta$].

## HYPERBOLA WITH CENTER $C(x_0, y_0)$ AND MAJOR AXIS PARALLEL TO $x$ AXIS

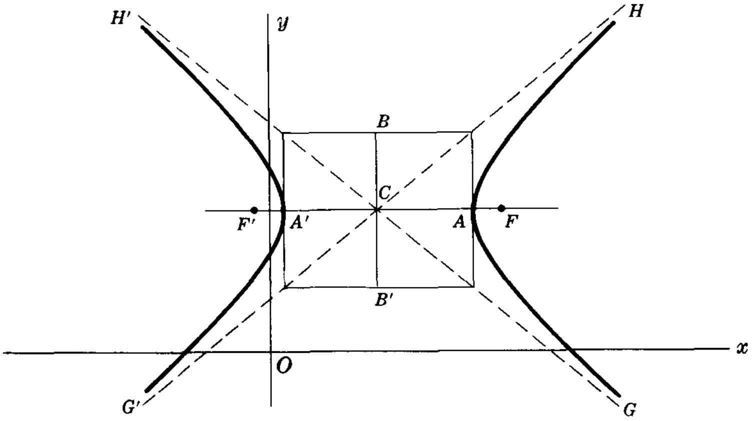


Fig. 10-17

**10.29** Length of major axis $A'A = 2a$

**10.30** Length of minor axis $B'B = 2b$

**10.31** Distance from center $C$ to focus $F$ or $F' = c = \sqrt{a^2 + b^2}$

**10.32** Eccentricity $\epsilon = \dfrac{c}{a} = \dfrac{\sqrt{a^2+b^2}}{a}$

**10.33** Equation in rectangular coordinates: $\dfrac{(x-x_0)^2}{a^2} - \dfrac{(y-y_0)^2}{b^2} = 1$

**10.34** Slopes of asymptotes $G'H$ and $GH' = \pm \dfrac{b}{a}$

**10.35** Equation in polar coordinates if $C$ is at $O$: $r^2 = \dfrac{a^2b^2}{b^2\cos^2\theta - a^2\sin^2\theta}$

**10.36** Equation in polar coordinates if $C$ is on $X$ axis and $F'$ is at $O$: $r = \dfrac{a(\epsilon^2 - 1)}{1 - \epsilon\cos\theta}$

**10.37** If $P$ is any point on the hyperbola, $PF - PF' = \pm 2a$ [depending on branch]

If the major axis is parallel to the $y$ axis, interchange $x$ and $y$ in the above or replace $\theta$ by $\frac{1}{2}\pi - \theta$ [or $90° - \theta$].

# 11 SPECIAL PLANE CURVES

## LEMNISCATE

**11.1** Equation in polar coordinates:

$$r^2 = a^2 \cos 2\theta$$

**11.2** Equation in rectangular coordinates:

$$(x^2+y^2)^2 = a^2(x^2-y^2)$$

**11.3** Angle between $AB'$ or $A'B$ and $x$ axis $= 45°$

**11.4** Area of one loop $= \frac{1}{2}a^2$

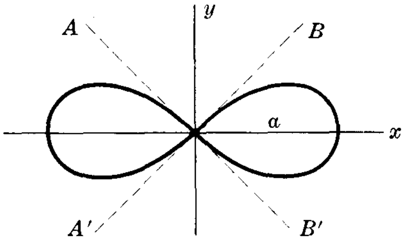


Fig. 11-1

## CYCLOID

**11.5** Equations in parametric form:

$$\begin{cases} x = a(\phi - \sin\phi) \\ y = a(1 - \cos\phi) \end{cases}$$

**11.6** Area of one arch $= 3\pi a^2$

**11.7** Arc length of one arch $= 8a$

This is a curve described by a point $P$ on a circle of radius $a$ rolling along $x$ axis.

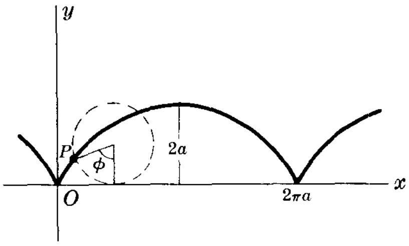


Fig. 11-2

## HYPOCYCLOID WITH FOUR CUSPS

**11.8** Equation in rectangular coordinates:

$$x^{2/3} + y^{2/3} = a^{2/3}$$

**11.9** Equations in parametric form:

$$\begin{cases} x = a\cos^3\theta \\ y = a\sin^3\theta \end{cases}$$

**11.10** Area bounded by curve $= \frac{3}{8}\pi a^2$

**11.11** Arc length of entire curve $= 6a$

This is a curve described by a point $P$ on a circle of radius $a/4$ as it rolls on the inside of a circle of radius $a$.

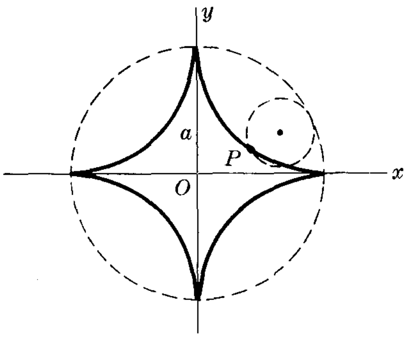
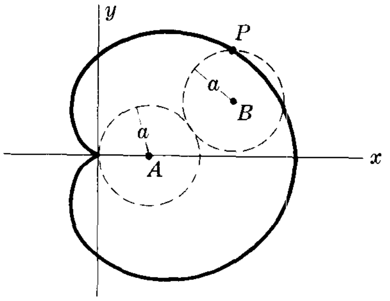


Fig. 11-3

## CARDIOID



**11.12** Equation: $r = a(1 + \cos\theta)$

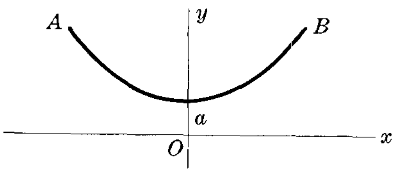
**11.13** Area bounded by curve $= \frac{3}{2}\pi a^2$

**11.14** Arc length of curve $= 8a$

This is the curve described by a point $P$ of a circle of radius $a$ as it rolls on the outside of a fixed circle of radius $a$. The curve is also a special case of the limacon of Pascal [see 11.32].

Fig. 11-4

## CATENARY

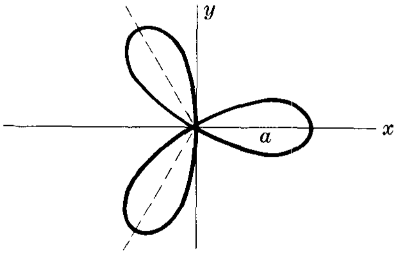


**11.15** Equation: $y = \frac{a}{2}(e^{x/a} + e^{-x/a}) = a\cosh\frac{x}{a}$

This is the curve in which a heavy uniform chain would hang if suspended vertically from fixed points $A$ and $B$.

Fig. 11-5

## THREE-LEAVED ROSE



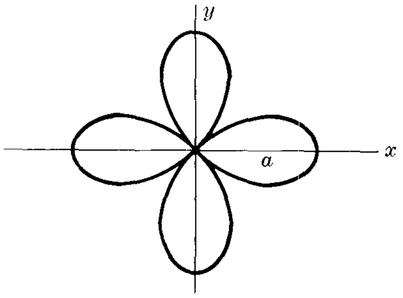
**11.16** Equation: $r = a\cos 3\theta$

The equation $r = a\sin 3\theta$ is a similar curve obtained by rotating the curve of Fig. 11-6 counterclockwise through 30° or $\pi/6$ radians.

In general $r = a\cos n\theta$ or $r = a\sin n\theta$ has $n$ leaves if $n$ is odd.

Fig. 11-6

## FOUR-LEAVED ROSE



**11.17** Equation: $r = a\cos 2\theta$

The equation $r = a\sin 2\theta$ is a similar curve obtained by rotating the curve of Fig. 11-7 counterclockwise through 45° or $\pi/4$ radians.

In general $r = a\cos n\theta$ or $r = a\sin n\theta$ has $2n$ leaves if $n$ is even.

Fig. 11-7

## EPICYCLOID

**11.18** Parametric equations:

$$\begin{cases} x = (a+b)\cos\theta - b\cos\left(\dfrac{a+b}{b}\right)\theta \\ y = (a+b)\sin\theta - b\sin\left(\dfrac{a+b}{b}\right)\theta \end{cases}$$

This is the curve described by a point $P$ on a circle of radius $b$ as it rolls on the outside of a circle of radius $a$.

The cardioid [Fig. 11-4] is a special case of an epicycloid.

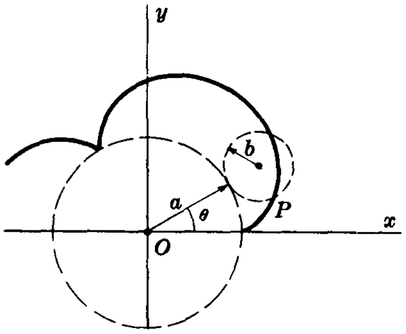


Fig. 11-8

## GENERAL HYPOCYCLOID

**11.19** Parametric equations:

$$\begin{cases} x = (a-b)\cos\phi + b\cos\left(\dfrac{a-b}{b}\right)\phi \\ y = (a-b)\sin\phi - b\sin\left(\dfrac{a-b}{b}\right)\phi \end{cases}$$

This is the curve described by a point $P$ on a circle of radius $b$ as it rolls on the inside of a circle of radius $a$.

If $b = a/4$, the curve is that of Fig. 11-3.

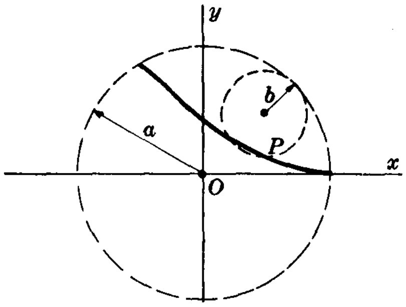


Fig. 11-9

## TROCHOID

**11.20** Parametric equations: $\begin{cases} x = a\phi - b\sin\phi \\ y = a - b\cos\phi \end{cases}$

This is the curve described by a point $P$ at distance $b$ from the center of a circle of radius $a$ as the circle rolls on the $x$ axis.

If $b < a$, the curve is as shown in Fig. 11-10 and is called a *curtate cycloid.*

If $b > a$, the curve is as shown in Fig. 11-11 and is called a *prolate cycloid.*

If $b = a$, the curve is the cycloid of Fig. 11-2.

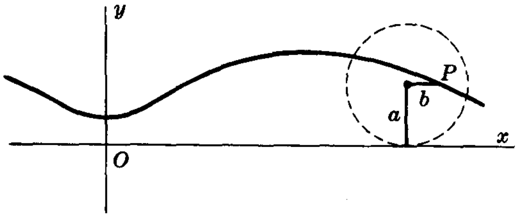


Fig. 11-10

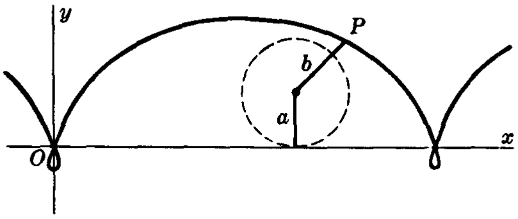


Fig. 11-11

## TRACTRIX

**11.21** Parametric equations: $\begin{cases} x = a(\ln\cot\frac{1}{2}\phi - \cos\phi) \\ y = a\sin\phi \end{cases}$

This is the curve described by endpoint $P$ of a taut string $PQ$ of length $a$ as the other end $Q$ is moved along the $x$ axis.

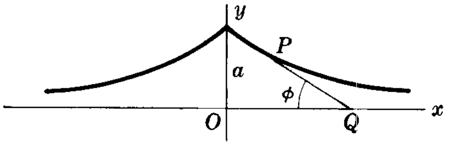


Fig. 11-12

## WITCH OF AGNESI

**11.22** Equation in rectangular coordinates: $y = \dfrac{8a^3}{x^2 + 4a^2}$

**11.23** Parametric equations: $\begin{cases} x = 2a\cot\theta \\ y = a(1 - \cos 2\theta) \end{cases}$

In Fig. 11-13 the variable line $OA$ intersects $y = 2a$ and the circle of radius $a$ with center $(0, a)$ at $A$ and $B$ respectively. Any point $P$ on the "witch" is located by constructing lines parallel to the $x$ and $y$ axes through $B$ and $A$ respectively and determining the point $P$ of intersection.

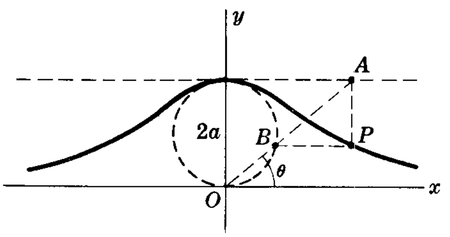


Fig. 11-13

## FOLIUM OF DESCARTES

**11.24** Equation in rectangular coordinates:

$$x^3 + y^3 = 3axy$$

**11.25** Parametric equations:

$$\begin{cases} x = \dfrac{3at}{1 + t^3} \\ y = \dfrac{3at^2}{1 + t^3} \end{cases}$$

**11.26** Area of loop $= \dfrac{3}{2}a^2$

**11.27** Equation of asymptote: $x + y + a = 0$

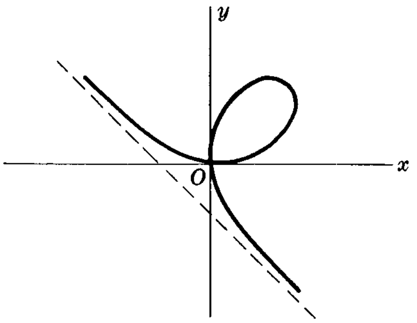


Fig. 11-14

## INVOLUTE OF A CIRCLE

**11.28** Parametric equations:

$$\begin{cases} x = a(\cos\phi + \phi\sin\phi) \\ y = a(\sin\phi - \phi\cos\phi) \end{cases}$$

This is the curve described by the endpoint $P$ of a string as it unwinds from a circle of radius $a$ while held taut.

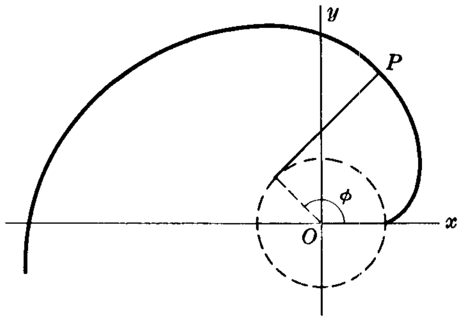


Fig. 11-15

## EVOLUTE OF AN ELLIPSE

**11.29** Equation in rectangular coordinates:

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$$

**11.30** Parametric equations:

$$\begin{cases} ax = (a^2 - b^2)\cos^3\theta \\ by = (a^2 - b^2)\sin^3\theta \end{cases}$$

This curve is the envelope of the normals to the ellipse $x^2/a^2 + y^2/b^2 = 1$ shown dashed in Fig. 11-16.

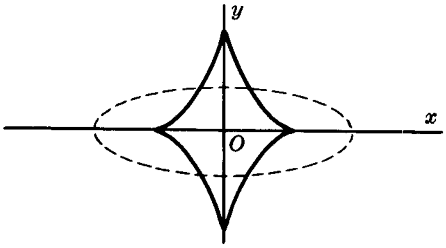


Fig. 11-16

## OVALS OF CASSINI

**11.31** Polar equation: $r^4 + a^4 - 2a^2r^2\cos 2\theta = b^4$

This is the curve described by a point $P$ such that the product of its distances from two fixed points [distance $2a$ apart] is a constant $b^2$.

The curve is as in Fig. 11-17 or Fig. 11-18 according as $b < a$ or $b > a$ respectively.

If $b = a$, the curve is a *lemniscate* [Fig. 11-1].

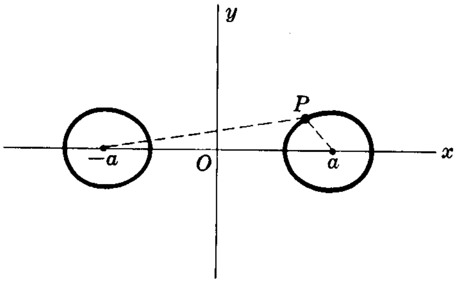


Fig. 11-17

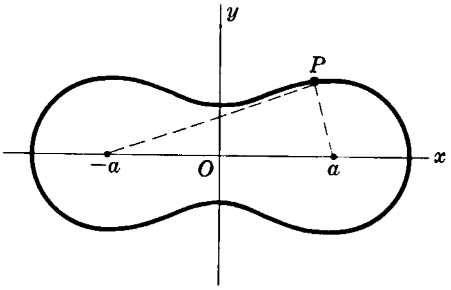


Fig. 11-18

## LIMACON OF PASCAL

**11.32** Polar equation: $r = b + a\cos\theta$

Let $OQ$ be a line joining origin $O$ to any point $Q$ on a circle of diameter $a$ passing through $O$. Then the curve is the locus of all points $P$ such that $PQ = b$.

The curve is as in Fig. 11-19 or Fig. 11-20 according as $b > a$ or $b < a$ respectively. If $b = a$, the curve is a *cardioid* [Fig. 11-4].

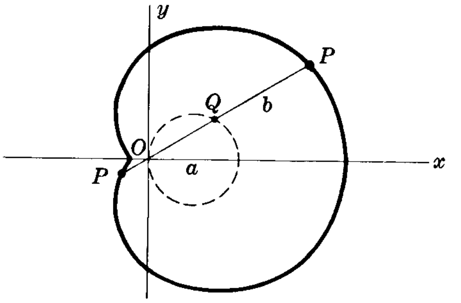


Fig. 11-19

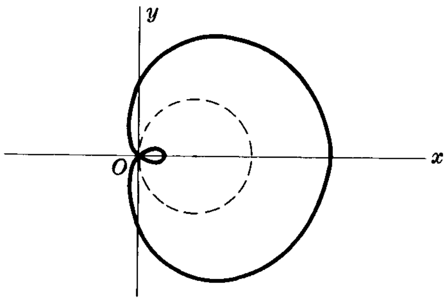


Fig. 11-20

## CISSOID OF DIOCLES

**11.33** Equation in rectangular coordinates:

$$y^2 = \frac{x^3}{2a - x}$$

**11.34** Parametric equations:

$$\begin{cases} x = 2a \sin^2 \theta \\ y = \dfrac{2a \sin^3 \theta}{\cos \theta} \end{cases}$$

This is the curve described by a point $P$ such that the distance $OP$ = distance $RS$. It is used in the problem of *duplication of a cube*, i.e. finding the side of a cube which has twice the volume of a given cube.

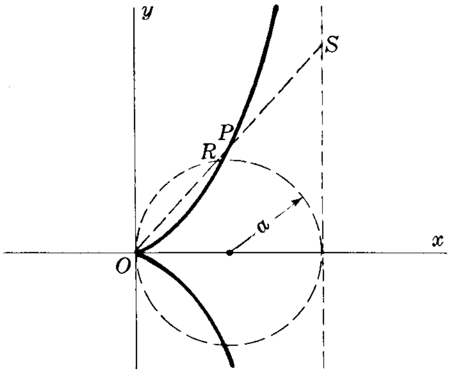


Fig. 11-21

## SPIRAL OF ARCHIMEDES

**11.35** Polar equation: $r = a\theta$

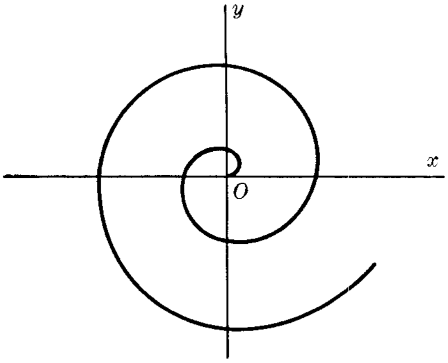


Fig. 11-22

# 12 FORMULAS from SOLID ANALYTIC GEOMETRY

## DISTANCE $d$ BETWEEN TWO POINTS $P_1(x_1, y_1, z_1)$ AND $P_2(x_2, y_2, z_2)$

**12.1** $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

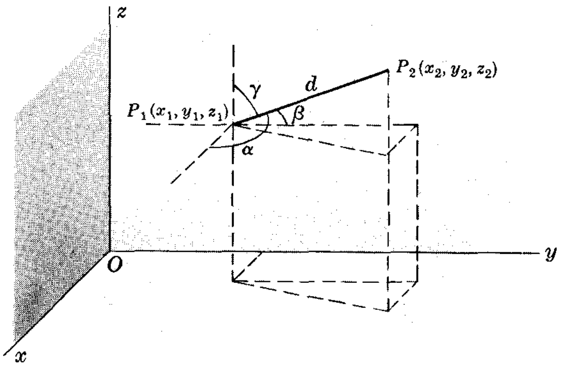


Fig. 12-1

## DIRECTION COSINES OF LINE JOINING POINTS $P_1(x_1, y_1, z_1)$ AND $P_2(x_2, y_2, z_2)$

**12.2** $$l = \cos\alpha = \frac{x_2 - x_1}{d}, \quad m = \cos\beta = \frac{y_2 - y_1}{d}, \quad n = \cos\gamma = \frac{z_2 - z_1}{d}$$

where $\alpha, \beta, \gamma$ are the angles which line $P_1P_2$ makes with the positive $x, y, z$ axes respectively and $d$ is given by 12.1 [see Fig. 12-1].

## RELATIONSHIP BETWEEN DIRECTION COSINES

**12.3** $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \quad \text{or} \quad l^2 + m^2 + n^2 = 1$$

## DIRECTION NUMBERS

Numbers $L, M, N$ which are proportional to the direction cosines $l, m, n$ are called *direction numbers.* The relationship between them is given by

**12.4** $$l = \frac{L}{\sqrt{L^2 + M^2 + N^2}}, \quad m = \frac{M}{\sqrt{L^2 + M^2 + N^2}}, \quad n = \frac{N}{\sqrt{L^2 + M^2 + N^2}}$$

## EQUATIONS OF LINE JOINING $P_1(x_1, y_1, z_1)$ AND $P_2(x_2, y_2, z_2)$ IN STANDARD FORM

**12.5** $$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} \quad \text{or} \quad \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

These are also valid if $l, m, n$ are replaced by $L, M, N$ respectively.

## EQUATIONS OF LINE JOINING $P_1(x_1, y_1, z_1)$ AND $P_2(x_2, y_2, z_2)$ IN PARAMETRIC FORM

**12.6** $$x = x_1 + lt, \quad y = y_1 + mt, \quad z = z_1 + nt$$

These are also valid if $l, m, n$ are replaced by $L, M, N$ respectively.

## ANGLE $\phi$ BETWEEN TWO LINES WITH DIRECTION COSINES $l_1, m_1, n_1$ AND $l_2, m_2, n_2$

**12.7** $$\cos\phi = l_1 l_2 + m_1 m_2 + n_1 n_2$$

## GENERAL EQUATION OF A PLANE

**12.8** $$Ax + By + Cz + D = 0 \qquad [A, B, C, D \text{ are constants}]$$

## EQUATION OF PLANE PASSING THROUGH POINTS $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, $(x_3, y_3, z_3)$

**12.9** $$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

or

**12.10** $$\begin{vmatrix} y_2-y_1 & z_2-z_1 \\ y_3-y_1 & z_3-z_1 \end{vmatrix}(x-x_1) + \begin{vmatrix} z_2-z_1 & x_2-x_1 \\ z_3-z_1 & x_3-x_1 \end{vmatrix}(y-y_1) + \begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{vmatrix}(z-z_1) = 0$$

## EQUATION OF PLANE IN INTERCEPT FORM

**12.11** $$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

where $a, b, c$ are the intercepts on the $x, y, z$ axes respectively.

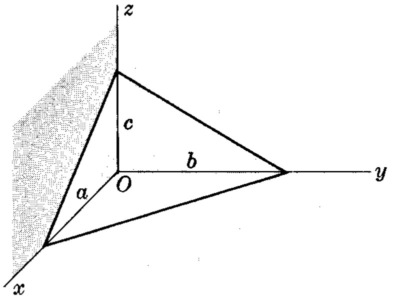


Fig. 12-2

## EQUATIONS OF LINE THROUGH $(x_0, y_0, z_0)$ AND PERPENDICULAR TO PLANE $Ax + By + Cz + D = 0$

**12.12** $$\frac{x - x_0}{A} = \frac{y - y_0}{B} = \frac{z - z_0}{C} \quad \text{or} \quad x = x_0 + At, \quad y = y_0 + Bt, \quad z = z_0 + Ct$$

Note that the direction numbers for a line perpendicular to the plane $Ax + By + Cz + D = 0$ are $A, B, C$.

## DISTANCE FROM POINT $(x_0, y_0, z_0)$ TO PLANE $Ax + By + Cz + D = 0$

**12.13** $$\frac{Ax_0 + By_0 + Cz_0 + D}{\pm\sqrt{A^2 + B^2 + C^2}}$$

where the sign is chosen so that the distance is nonnegative.

## NORMAL FORM FOR EQUATION OF PLANE

**12.14** $$x \cos\alpha + y \cos\beta + z \cos\gamma = p$$

where $p$ = perpendicular distance from $O$ to plane at $P$ and $\alpha, \beta, \gamma$ are angles between $OP$ and positive $x, y, z$ axes.

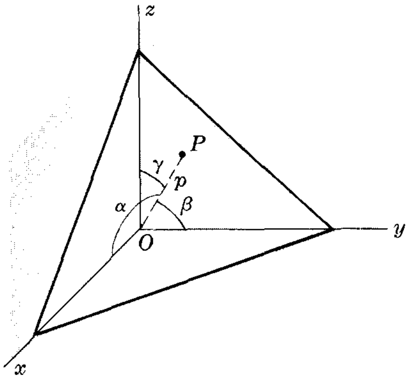


Fig. 12-3

## TRANSFORMATION OF COORDINATES INVOLVING PURE TRANSLATION

**12.15** $$\begin{cases} x = x' + x_0 \\ y = y' + y_0 \\ z = z' + z_0 \end{cases} \quad \text{or} \quad \begin{cases} x' = x - x_0 \\ y' = y - y_0 \\ z' = z - z_0 \end{cases}$$

where $(x, y, z)$ are old coordinates [i.e. coordinates relative to $xyz$ system], $(x', y', z')$ are new coordinates [relative to $x'y'z'$ system] and $(x_0, y_0, z_0)$ are the coordinates of the new origin $O'$ relative to the old $xyz$ coordinate system.

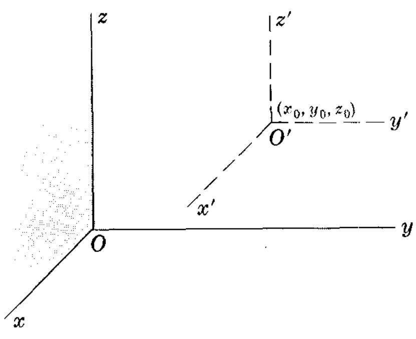


Fig. 12-4

## TRANSFORMATION OF COORDINATES INVOLVING PURE ROTATION

**12.16**
$$\begin{cases} x = l_1x' + l_2y' + l_3z' \\ y = m_1x' + m_2y' + m_3z' \\ z = n_1x' + n_2y' + n_3z' \end{cases}$$

or
$$\begin{cases} x' = l_1x + m_1y + n_1z \\ y' = l_2x + m_2y + n_2z \\ z' = l_3x + m_3y + n_3z \end{cases}$$

where the origins of the $xyz$ and $x'y'z'$ systems are the same and $l_1, m_1, n_1$; $l_2, m_2, n_2$; $l_3, m_3, n_3$ are the direction cosines of the $x', y', z'$ axes relative to the $x, y, z$ axes respectively.

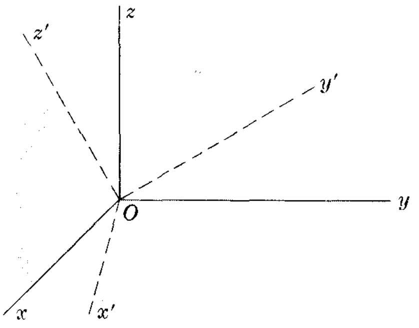


Fig. 12-5

## TRANSFORMATION OF COORDINATES INVOLVING TRANSLATION AND ROTATION

**12.17**
$$\begin{cases} x = l_1x' + l_2y' + l_3z' + x_0 \\ y = m_1x' + m_2y' + m_3z' + y_0 \\ z = n_1x' + n_2y' + n_3z' + z_0 \end{cases}$$

or
$$\begin{cases} x' = l_1(x-x_0) + m_1(y-y_0) + n_1(z-z_0) \\ y' = l_2(x-x_0) + m_2(y-y_0) + n_2(z-z_0) \\ z' = l_3(x-x_0) + m_3(y-y_0) + n_3(z-z_0) \end{cases}$$

where the origin $O'$ of the $x'y'z'$ system has coordinates $(x_0, y_0, z_0)$ relative to the $xyz$ system and $l_1, m_1, n_1$; $l_2, m_2, n_2$; $l_3, m_3, n_3$ are the direction cosines of the $x', y', z'$ axes relative to the $x, y, z$ axes respectively.

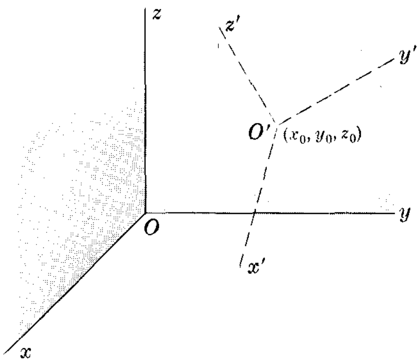


Fig. 12-6

## CYLINDRICAL COORDINATES $(r, \theta, z)$

A point $P$ can be located by cylindrical coordinates $(r, \theta, z)$ [see Fig. 12-7] as well as rectangular coordinates $(x, y, z)$.

The transformation between these coordinates is

**12.18**
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \quad \text{or} \quad \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}(y/x) \\ z = z \end{cases}$$

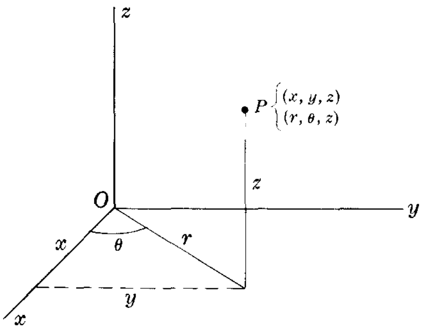


Fig. 12-7

## SPHERICAL COORDINATES $(r, \theta, \phi)$

A point $P$ can be located by spherical coordinates $(r, \theta, \phi)$ [see Fig. 12-8] as well as rectangular coordinates $(x, y, z)$.

The transformation between those coordinates is

**12.19**
$$\begin{cases} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r\cos\theta \end{cases}$$

or
$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \phi = \tan^{-1}(y/x) \\ \theta = \cos^{-1}(z/\sqrt{x^2+y^2+z^2}) \end{cases}$$

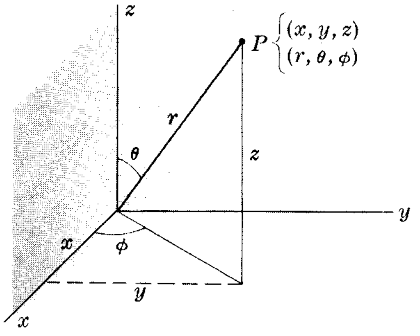


Fig. 12-8

## EQUATION OF SPHERE IN RECTANGULAR COORDINATES

**12.20** $$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = R^2$$

where the sphere has center $(x_0, y_0, z_0)$ and radius $R$.

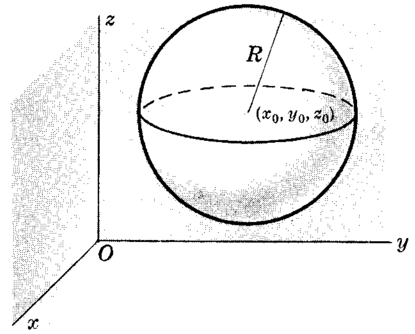


Fig. 12-9

## EQUATION OF SPHERE IN CYLINDRICAL COORDINATES

**12.21** $$r^2 - 2r_0 r\cos(\theta-\theta_0) + r_0^2 + (z-z_0)^2 = R^2$$

where the sphere has center $(r_0, \theta_0, z_0)$ in cylindrical coordinates and radius $R$.

If the center is at the origin the equation is

**12.22** $$r^2 + z^2 = R^2$$

## EQUATION OF SPHERE IN SPHERICAL COORDINATES

**12.23** $$r^2 + r_0^2 - 2r_0 r \sin\theta \sin\theta_0 \cos(\phi - \phi_0) = R^2$$

where the sphere has center $(r_0, \theta_0, \phi_0)$ in spherical coordinates and radius $R$.

If the center is at the origin the equation is

**12.24** $$r = R$$

## EQUATION OF ELLIPSOID WITH CENTER $(x_0, y_0, z_0)$ AND SEMI-AXES $a, b, c$

**12.25** $$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} + \frac{(z-z_0)^2}{c^2} = 1$$

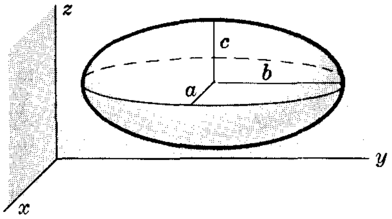


Fig. 12-10

## ELLIPTIC CYLINDER WITH AXIS AS $z$ AXIS

**12.26** $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a, b$ are semi-axes of elliptic cross section.

If $b = a$ it becomes a circular cylinder of radius $a$.

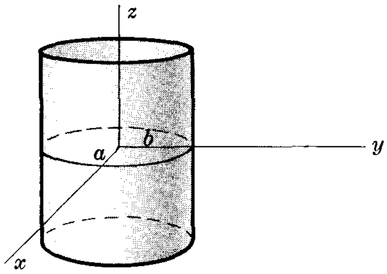


Fig. 12-11

## ELLIPTIC CONE WITH AXIS AS $z$ AXIS

**12.27** $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

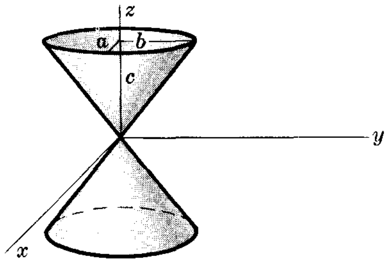


Fig. 12-12

## HYPERBOLOID OF ONE SHEET

**12.28** $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

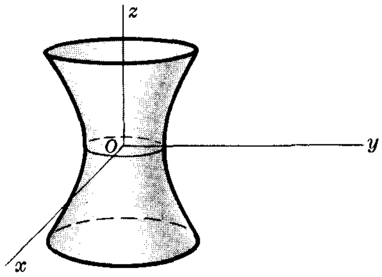


Fig. 12-13

## HYPERBOLOID OF TWO SHEETS

**12.29** $$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Note orientation of axes in Fig. 12-14.

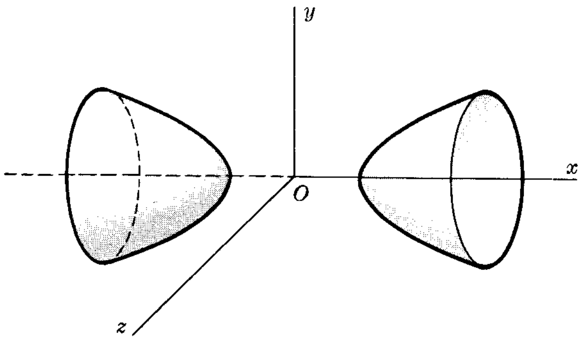


Fig. 12-14

## ELLIPTIC PARABOLOID

**12.30** $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

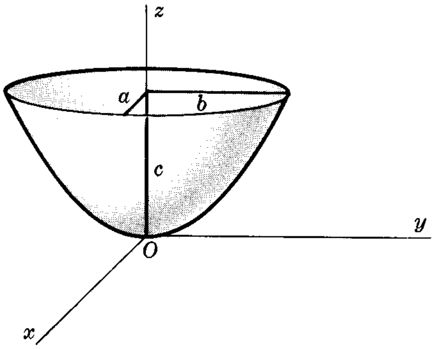


Fig. 12-15

## HYPERBOLIC PARABOLOID

**12.31** $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$$

Note orientation of axes in Fig. 12-16.

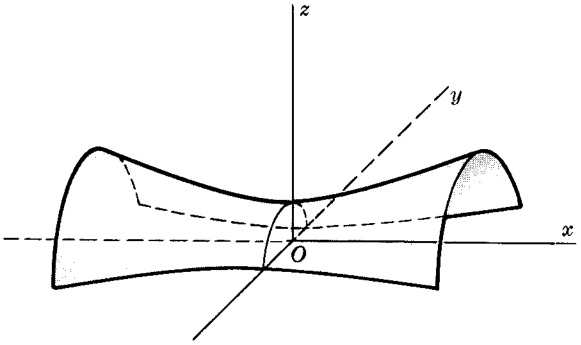


Fig. 12-16

# 13 DERIVATIVES

## DEFINITION OF A DERIVATIVE

If $y = f(x)$, the derivative of $y$ or $f(x)$ with respect to $x$ is defined as

**13.1** $$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

where $h = \Delta x$. The derivative is also denoted by $y'$, $df/dx$ or $f'(x)$. The process of taking a derivative is called *differentiation.*

## GENERAL RULES OF DIFFERENTIATION

In the following, $u, v, w$ are functions of $x$; $a, b, c, n$ are constants [restricted if indicated]; $e = 2.71828\ldots$ is the natural base of logarithms; $\ln u$ is the natural logarithm of $u$ [i.e. the logarithm to the base $e$] where it is assumed that $u > 0$ and all angles are in radians.

**13.2** $$\frac{d}{dx}(c) = 0$$

**13.3** $$\frac{d}{dx}(cx) = c$$

**13.4** $$\frac{d}{dx}(cx^n) = ncx^{n-1}$$

**13.5** $$\frac{d}{dx}(u \pm v \pm w \pm \cdots) = \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx} \pm \cdots$$

**13.6** $$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

**13.7** $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

**13.8** $$\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$$

**13.9** $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v(du/dx) - u(dv/dx)}{v^2}$$

**13.10** $$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

**13.11** $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad \text{(Chain rule)}$$

**13.12** $$\frac{du}{dx} = \frac{1}{dx/du}$$

**13.13** $$\frac{dy}{dx} = \frac{dy/du}{dx/du}$$

## DERIVATIVES OF TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS

**13.14** $$\frac{d}{dx}\sin u = \cos u\frac{du}{dx}$$

**13.15** $$\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}$$

**13.16** $$\frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx}$$

**13.17** $$\frac{d}{dx}\cot u = -\csc^2 u\frac{du}{dx}$$

**13.18** $$\frac{d}{dx}\sec u = \sec u\tan u\frac{du}{dx}$$

**13.19** $$\frac{d}{dx}\csc u = -\csc u\cot u\frac{du}{dx}$$

**13.20** $$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad \left[-\frac{\pi}{2} < \sin^{-1} u < \frac{\pi}{2}\right]$$

**13.21** $$\frac{d}{dx}\cos^{-1} u = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad [0 < \cos^{-1} u < \pi]$$

**13.22** $$\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx} \qquad \left[-\frac{\pi}{2} < \tan^{-1} u < \frac{\pi}{2}\right]$$

**13.23** $$\frac{d}{dx}\cot^{-1} u = \frac{-1}{1+u^2}\frac{du}{dx} \qquad [0 < \cot^{-1} u < \pi]$$

**13.24** $$\frac{d}{dx}\sec^{-1} u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} = \frac{\pm 1}{u\sqrt{u^2-1}}\frac{du}{dx} \qquad \left[\begin{array}{l} + \text{ if } 0 < \sec^{-1} u < \pi/2 \\ - \text{ if } \pi/2 < \sec^{-1} u < \pi \end{array}\right]$$

**13.25** $$\frac{d}{dx}\csc^{-1} u = \frac{-1}{|u|\sqrt{u^2-1}}\frac{du}{dx} = \frac{\mp 1}{u\sqrt{u^2-1}}\frac{du}{dx} \qquad \left[\begin{array}{l} - \text{ if } 0 < \csc^{-1} u < \pi/2 \\ + \text{ if } -\pi/2 < \csc^{-1} u < 0 \end{array}\right]$$

## DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

**13.26** $$\frac{d}{dx}\log_a u = \frac{\log_a e}{u}\frac{du}{dx} \qquad a \neq 0, 1$$

**13.27** $$\frac{d}{dx}\ln u = \frac{d}{dx}\log_e u = \frac{1}{u}\frac{du}{dx}$$

**13.28** $$\frac{d}{dx}a^u = a^u \ln a\frac{du}{dx}$$

**13.29** $$\frac{d}{dx}e^u = e^u\frac{du}{dx}$$

**13.30** $$\frac{d}{dx}u^v = \frac{d}{dx}e^{v\ln u} = e^{v\ln u}\frac{d}{dx}[v\ln u] = vu^{v-1}\frac{du}{dx} + u^v\ln u\frac{dv}{dx}$$

## DERIVATIVES OF HYPERBOLIC AND INVERSE HYPERBOLIC FUNCTIONS

**13.31** $$\frac{d}{dx}\sinh u = \cosh u\frac{du}{dx}$$

**13.32** $$\frac{d}{dx}\cosh u = \sinh u\frac{du}{dx}$$

**13.33** $$\frac{d}{dx}\tanh u = \operatorname{sech}^2 u\frac{du}{dx}$$

**13.34** $$\frac{d}{dx}\coth u = -\operatorname{csch}^2 u\frac{du}{dx}$$

**13.35** $$\frac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u\tanh u\frac{du}{dx}$$

**13.36** $$\frac{d}{dx}\operatorname{csch} u = -\operatorname{csch} u\coth u\frac{du}{dx}$$

**13.37** $$\frac{d}{dx}\sinh^{-1}u = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}$$

**13.38** $$\frac{d}{dx}\cosh^{-1}u = \frac{\pm 1}{\sqrt{u^2-1}}\frac{du}{dx} \qquad \left[\begin{array}{l} + \text{ if } \cosh^{-1}u > 0,\ u > 1 \\ - \text{ if } \cosh^{-1}u < 0,\ u > 1 \end{array}\right]$$

**13.39** $$\frac{d}{dx}\tanh^{-1}u = \frac{1}{1-u^2}\frac{du}{dx} \qquad [-1 < u < 1]$$

**13.40** $$\frac{d}{dx}\coth^{-1}u = \frac{1}{1-u^2}\frac{du}{dx} \qquad [u > 1 \text{ or } u < -1]$$

**13.41** $$\frac{d}{dx}\operatorname{sech}^{-1}u = \frac{\mp 1}{u\sqrt{1-u^2}}\frac{du}{dx} \qquad \left[\begin{array}{l} - \text{ if } \operatorname{sech}^{-1}u > 0,\ 0 < u < 1 \\ + \text{ if } \operatorname{sech}^{-1}u < 0,\ 0 < u < 1 \end{array}\right]$$

**13.42** $$\frac{d}{dx}\operatorname{csch}^{-1}u = \frac{-1}{|u|\sqrt{1+u^2}}\frac{du}{dx} = \frac{\mp 1}{u\sqrt{1+u^2}}\frac{du}{dx} \qquad [- \text{ if } u > 0,\ + \text{ if } u < 0]$$

## HIGHER DERIVATIVES

The second, third and higher derivatives are defined as follows.

**13.43** $$\text{Second derivative} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x) = y''$$

**13.44** $$\text{Third derivative} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = f'''(x) = y'''$$

**13.45** $$n\text{th derivative} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} = f^{(n)}(x) = y^{(n)}$$

## LEIBNITZ'S RULE FOR HIGHER DERIVATIVES OF PRODUCTS

Let $D^p$ stand for the operator $\frac{d^p}{dx^p}$ so that $D^pu = \frac{d^pu}{dx^p}$ = the $p$th derivative of $u$. Then

**13.46** $$D^n(uv) = uD^nv + \binom{n}{1}(Du)(D^{n-1}v) + \binom{n}{2}(D^2u)(D^{n-2}v) + \cdots + vD^nu$$

where $\binom{n}{1}, \binom{n}{2}, \ldots$ are the binomial coefficients [page 3].

As special cases we have

**13.47** $$\frac{d^2}{dx^2}(uv) = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

**13.48** $$\frac{d^3}{dx^3}(uv) = u\frac{d^3v}{dx^3} + 3\frac{du}{dx}\frac{d^2v}{dx^2} + 3\frac{d^2u}{dx^2}\frac{dv}{dx} + v\frac{d^3u}{dx^3}$$

## DIFFERENTIALS

Let $y = f(x)$ and $\Delta y = f(x+\Delta x) - f(x)$. Then

**13.49** $$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x) + \epsilon = \frac{dy}{dx} + \epsilon$$

where $\epsilon \to 0$ as $\Delta x \to 0$. Thus

**13.50** $$\Delta y = f'(x)\,\Delta x + \epsilon\,\Delta x$$

If we call $\Delta x = dx$ the differential of $x$, then we define the differential of $y$ to be

**13.51** $$dy = f'(x)\,dx$$

## RULES FOR DIFFERENTIALS

The rules for differentials are exactly analogous to those for derivatives. As examples we observe that

**13.52** $$d(u \pm v \pm w \pm \cdots) = du \pm dv \pm dw \pm \cdots$$

**13.53** $$d(uv) = u\,dv + v\,du$$

**13.54** $$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$$

**13.55** $$d(u^n) = nu^{n-1}\,du$$

**13.56** $$d(\sin u) = \cos u\,du$$

**13.57** $$d(\cos u) = -\sin u\,du$$

## PARTIAL DERIVATIVES

Let $f(x,y)$ be a function of the two variables $x$ and $y$. Then we define the partial derivative of $f(x,y)$ with respect to $x$, keeping $y$ constant, to be

**13.58** $$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x,\,y) - f(x,y)}{\Delta x}$$

Similarly the partial derivative of $f(x,y)$ with respect to $y$, keeping $x$ constant, is defined to be

**13.59** $$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x,\,y+\Delta y) - f(x,y)}{\Delta y}$$

Partial derivatives of higher order can be defined as follows.

**13.60** $$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

**13.61** $$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \qquad \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

The results in 13.61 will be equal if the function and its partial derivatives are continuous, i.e. in such case the order of differentiation makes no difference.

The differential of $f(x,y)$ is defined as

**13.62** $$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

where $dx = \Delta x$ and $dy = \Delta y$.

Extension to functions of more than two variables are exactly analogous.

# 14

# INDEFINITE INTEGRALS

## DEFINITION OF AN INDEFINITE INTEGRAL

If $\frac{dy}{dx} = f(x)$, then $y$ is the function whose derivative is $f(x)$ and is called the *anti-derivative* of $f(x)$ or the *indefinite integral* of $f(x)$, denoted by $\int f(x)\,dx$. Similarly if $y = \int f(u)\,du$, then $\frac{dy}{du} = f(u)$. Since the derivative of a constant is zero, all indefinite integrals differ by an arbitrary constant.

For the definition of a definite integral, see page 94. The process of finding an integral is called *integration.*

## GENERAL RULES OF INTEGRATION

In the following, $u, v, w$ are functions of $x$; $a, b, p, q, n$ any constants, restricted if indicated; $e = 2.71828\ldots$ is the natural base of logarithms; $\ln u$ denotes the natural logarithm of $u$ where it is assumed that $u > 0$ [in general, to extend formulas to cases where $u < 0$ as well, replace $\ln u$ by $\ln|u|$]; all angles are in radians; all constants of integration are omitted but implied.

**14.1** $$\int a\,dx = ax$$

**14.2** $$\int af(x)\,dx = a\int f(x)\,dx$$

**14.3** $$\int (u \pm v \pm w \pm \cdots)\,dx = \int u\,dx \pm \int v\,dx \pm \int w\,dx \pm \cdots$$

**14.4** $$\int u\,dv = uv - \int v\,du \qquad \text{[Integration by parts]}$$

For generalized integration by parts, see 14.48.

**14.5** $$\int f(ax)\,dx = \frac{1}{a}\int f(u)\,du$$

**14.6** $$\int F\{f(x)\}\,dx = \int F(u)\frac{dx}{du}\,du = \int \frac{F(u)}{f'(x)}\,du \quad \text{where } u = f(x)$$

**14.7** $$\int u^n\,du = \frac{u^{n+1}}{n+1}, \quad n \neq -1 \qquad [\text{For } n = -1, \text{ see } 14.8]$$

**14.8** $$\begin{aligned}\int \frac{du}{u} &= \ln u \quad \text{if } u > 0 \text{ or } \ln(-u) \text{ if } u < 0\\ &= \ln|u|\end{aligned}$$

**14.9** $$\int e^u\,du = e^u$$

**14.10** $$\int a^u\,du = \int e^{u\ln a}\,du = \frac{e^{u\ln a}}{\ln a} = \frac{a^u}{\ln a}, \quad a > 0,\ a \neq 1$$

**14.11** $$\int \sin u\,du = -\cos u$$

**14.12** $$\int \cos u\,du = \sin u$$

**14.13** $$\int \tan u\,du = \ln \sec u = -\ln \cos u$$

**14.14** $$\int \cot u\,du = \ln \sin u$$

**14.15** $$\int \sec u\,du = \ln(\sec u + \tan u) = \ln \tan\left(\frac{u}{2}+\frac{\pi}{4}\right)$$

**14.16** $$\int \csc u\,du = \ln(\csc u - \cot u) = \ln \tan\frac{u}{2}$$

**14.17** $$\int \sec^2 u\,du = \tan u$$

**14.18** $$\int \csc^2 u\,du = -\cot u$$

**14.19** $$\int \tan^2 u\,du = \tan u - u$$

**14.20** $$\int \cot^2 u\,du = -\cot u - u$$

**14.21** $$\int \sin^2 u\,du = \frac{u}{2} - \frac{\sin 2u}{4} = \tfrac{1}{2}(u - \sin u \cos u)$$

**14.22** $$\int \cos^2 u\,du = \frac{u}{2} + \frac{\sin 2u}{4} = \tfrac{1}{2}(u + \sin u \cos u)$$

**14.23** $$\int \sec u \tan u\,du = \sec u$$

**14.24** $$\int \csc u \cot u\,du = -\csc u$$

**14.25** $$\int \sinh u\,du = \cosh u$$

**14.26** $$\int \cosh u\,du = \sinh u$$

**14.27** $$\int \tanh u\,du = \ln \cosh u$$

**14.28** $$\int \coth u\,du = \ln \sinh u$$

**14.29** $$\int \operatorname{sech} u\,du = \sin^{-1}(\tanh u) \quad \text{or} \quad 2\tan^{-1} e^u$$

**14.30** $$\int \operatorname{csch} u\,du = \ln \tanh\frac{u}{2} \quad \text{or} \quad -\coth^{-1} e^u$$

**14.31** $$\int \operatorname{sech}^2 u\,du = \tanh u$$

**14.32** $$\int \operatorname{csch}^2 u\,du = -\coth u$$

**14.33** $$\int \tanh^2 u\,du = u - \tanh u$$

**14.34** $\int \coth^2 u\, du \;=\; u - \coth u$

**14.35** $\int \sinh^2 u\, du \;=\; \frac{\sinh 2u}{4} - \frac{u}{2} \;=\; \tfrac{1}{2}(\sinh u \cosh u - u)$

**14.36** $\int \cosh^2 u\, du \;=\; \frac{\sinh 2u}{4} + \frac{u}{2} \;=\; \tfrac{1}{2}(\sinh u \cosh u + u)$

**14.37** $\int \operatorname{sech} u \tanh u\, du \;=\; -\operatorname{sech} u$

**14.38** $\int \operatorname{csch} u \coth u\, du \;=\; -\operatorname{csch} u$

**14.39** $\int \frac{du}{u^2 + a^2} \;=\; \frac{1}{a}\tan^{-1}\frac{u}{a}$

**14.40** $\int \frac{du}{u^2 - a^2} \;=\; \frac{1}{2a}\ln\left(\frac{u-a}{u+a}\right) \;=\; -\frac{1}{a}\coth^{-1}\frac{u}{a} \qquad u^2 > a^2$

**14.41** $\int \frac{du}{a^2 - u^2} \;=\; \frac{1}{2a}\ln\left(\frac{a+u}{a-u}\right) \;=\; \frac{1}{a}\tanh^{-1}\frac{u}{a} \qquad u^2 < a^2$

**14.42** $\int \frac{du}{\sqrt{a^2 - u^2}} \;=\; \sin^{-1}\frac{u}{a}$

**14.43** $\int \frac{du}{\sqrt{u^2 + a^2}} \;=\; \ln\left(u + \sqrt{u^2 + a^2}\right) \quad \text{or} \quad \sinh^{-1}\frac{u}{a}$

**14.44** $\int \frac{du}{\sqrt{u^2 - a^2}} \;=\; \ln\left(u + \sqrt{u^2 - a^2}\right)$

**14.45** $\int \frac{du}{u\sqrt{u^2 - a^2}} \;=\; \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right|$

**14.46** $\int \frac{du}{u\sqrt{u^2 + a^2}} \;=\; -\frac{1}{a}\ln\left(\frac{a + \sqrt{u^2 + a^2}}{u}\right)$

**14.47** $\int \frac{du}{u\sqrt{a^2 - u^2}} \;=\; -\frac{1}{a}\ln\left(\frac{a + \sqrt{a^2 - u^2}}{u}\right)$

**14.48** $\int f^{(n)} g\, dx \;=\; f^{(n-1)}g - f^{(n-2)}g' + f^{(n-3)}g'' - \cdots (-1)^n \int f g^{(n)}\, dx$

This is called *generalized integration* by parts.

## IMPORTANT TRANSFORMATIONS

Often in practice an integral can be simplified by using an appropriate transformation or substitution and formula 14.6, page 57. The following list gives some transformations and their effects.

**14.49** $\int F(ax + b)\, dx \;=\; \frac{1}{a}\int F(u)\, du \qquad \text{where } u = ax + b$

**14.50** $\int F(\sqrt{ax + b})\, dx \;=\; \frac{2}{a}\int u\,F(u)\, du \qquad \text{where } u = \sqrt{ax + b}$

**14.51** $\int F(\sqrt[n]{ax + b})\, dx \;=\; \frac{n}{a}\int u^{n-1}F(u)\, du \qquad \text{where } u = \sqrt[n]{ax + b}$

**14.52** $\int F(\sqrt{a^2 - x^2})\, dx \;=\; a\int F(a\cos u)\cos u\, du \qquad \text{where } x = a\sin u$

**14.53** $\int F(\sqrt{x^2 + a^2})\, dx \;=\; a\int F(a\sec u)\sec^2 u\, du \qquad \text{where } x = a\tan u$

**14.54** $\int F(\sqrt{x^2-a^2})\,dx = a\int F(a\tan u)\sec u\tan u\,du$ where $x = a\sec u$

**14.55** $\int F(e^{ax})\,dx = \frac{1}{a}\int \frac{F(u)}{u}\,du$ where $u = e^{ax}$

**14.56** $\int F(\ln x)\,dx = \int F(u)\,e^u\,du$ where $u = \ln x$

**14.57** $\int F\left(\sin^{-1}\frac{x}{a}\right)dx = a\int F(u)\cos u\,du$ where $u = \sin^{-1}\frac{x}{a}$

Similar results apply for other inverse trigonometric functions.

**14.58** $\int F(\sin x, \cos x)\,dx = 2\int F\left(\frac{2u}{1+u^2}, \frac{1-u^2}{1+u^2}\right)\frac{du}{1+u^2}$ where $u = \tan\frac{x}{2}$

## SPECIAL INTEGRALS

Pages 60 through 93 provide a table of integrals classified under special types. The remarks given on page 57 apply here as well. It is assumed in all cases that division by zero is excluded.

## INTEGRALS INVOLVING $ax + b$

**14.59** $\int \frac{dx}{ax+b} = \frac{1}{a}\ln(ax+b)$

**14.60** $\int \frac{x\,dx}{ax+b} = \frac{x}{a} - \frac{b}{a^2}\ln(ax+b)$

**14.61** $\int \frac{x^2\,dx}{ax+b} = \frac{(ax+b)^2}{2a^3} - \frac{2b(ax+b)}{a^3} + \frac{b^2}{a^3}\ln(ax+b)$

**14.62** $\int \frac{x^3\,dx}{ax+b} = \frac{(ax+b)^3}{3a^4} - \frac{3b(ax+b)^2}{2a^4} + \frac{3b^2(ax+b)}{a^4} - \frac{b^3}{a^4}\ln(ax+b)$

**14.63** $\int \frac{dx}{x(ax+b)} = \frac{1}{b}\ln\left(\frac{x}{ax+b}\right)$

**14.64** $\int \frac{dx}{x^2(ax+b)} = -\frac{1}{bx} + \frac{a}{b^2}\ln\left(\frac{ax+b}{x}\right)$

**14.65** $\int \frac{dx}{x^3(ax+b)} = \frac{2ax-b}{2b^2x^2} + \frac{a^2}{b^3}\ln\left(\frac{x}{ax+b}\right)$

**14.66** $\int \frac{dx}{(ax+b)^2} = \frac{-1}{a(ax+b)}$

**14.67** $\int \frac{x\,dx}{(ax+b)^2} = \frac{b}{a^2(ax+b)} + \frac{1}{a^2}\ln(ax+b)$

**14.68** $\int \frac{x^2\,dx}{(ax+b)^2} = \frac{ax+b}{a^3} - \frac{b^2}{a^3(ax+b)} - \frac{2b}{a^3}\ln(ax+b)$

**14.69** $\int \frac{x^3\,dx}{(ax+b)^2} = \frac{(ax+b)^2}{2a^4} - \frac{3b(ax+b)}{a^4} + \frac{b^3}{a^4(ax+b)} + \frac{3b^2}{a^4}\ln(ax+b)$

**14.70** $\int \frac{dx}{x(ax+b)^2} = \frac{1}{b(ax+b)} + \frac{1}{b^2}\ln\left(\frac{x}{ax+b}\right)$

**14.71** $\int \frac{dx}{x^2(ax+b)^2} = \frac{-a}{b^2(ax+b)} - \frac{1}{b^2x} + \frac{2a}{b^3}\ln\left(\frac{ax+b}{x}\right)$

**14.72** $$\int \frac{dx}{x^3(ax+b)^2} = -\frac{(ax+b)^2}{2b^4x^2} + \frac{3a(ax+b)}{b^4x} - \frac{a^3x}{b^4(ax+b)} - \frac{3a^2}{b^4}\ln\left(\frac{ax+b}{x}\right)$$

**14.73** $$\int \frac{dx}{(ax+b)^3} = \frac{-1}{2(ax+b)^2}$$

**14.74** $$\int \frac{x\,dx}{(ax+b)^3} = \frac{-1}{a^2(ax+b)} + \frac{b}{2a^2(ax+b)^2}$$

**14.75** $$\int \frac{x^2\,dx}{(ax+b)^3} = \frac{2b}{a^3(ax+b)} - \frac{b^2}{2a^3(ax+b)^2} + \frac{1}{a^3}\ln(ax+b)$$

**14.76** $$\int \frac{x^3\,dx}{(ax+b)^3} = \frac{x}{a^3} - \frac{3b^2}{a^4(ax+b)} + \frac{b^3}{2a^4(ax+b)^2} - \frac{3b}{a^4}\ln(ax+b)$$

**14.77** $$\int \frac{dx}{x(ax+b)^3} = \frac{a^2x^2}{2b^3(ax+b)^2} - \frac{2ax}{b^3(ax+b)} - \frac{1}{b^3}\ln\left(\frac{ax+b}{x}\right)$$

**14.78** $$\int \frac{dx}{x^2(ax+b)^3} = \frac{-a}{2b^2(ax+b)^2} - \frac{2a}{b^3(ax+b)} - \frac{1}{b^3x} + \frac{3a}{b^4}\ln\left(\frac{ax+b}{x}\right)$$

**14.79** $$\int \frac{dx}{x^3(ax+b)^3} = \frac{a^4x^2}{2b^5(ax+b)^2} - \frac{4a^3x}{b^5(ax+b)} - \frac{(ax+b)^2}{2b^5x^2} - \frac{6a^2}{b^5}\ln\left(\frac{ax+b}{x}\right)$$

**14.80** $$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(n+1)a}.$$ If $n=-1$, see 14.59.

**14.81** $$\int x(ax+b)^n\,dx = \frac{(ax+b)^{n+2}}{(n+2)a^2} - \frac{b(ax+b)^{n+1}}{(n+1)a^2}, \quad n \neq -1, -2$$

If $n = -1, -2$, see 14.60, 14.67.

**14.82** $$\int x^2(ax+b)^n\,dx = \frac{(ax+b)^{n+3}}{(n+3)a^3} - \frac{2b(ax+b)^{n+2}}{(n+2)a^3} + \frac{b^2(ax+b)^{n+1}}{(n+1)a^3}$$

If $n = -1, -2, -3$, see 14.61, 14.68, 14.75.

**14.83** $$\int x^m(ax+b)^n\,dx = \begin{cases} \dfrac{x^{m+1}(ax+b)^n}{m+n+1} + \dfrac{nb}{m+n+1}\displaystyle\int x^m(ax+b)^{n-1}\,dx \\[2ex] \dfrac{x^m(ax+b)^{n+1}}{(m+n+1)a} - \dfrac{mb}{(m+n+1)a}\displaystyle\int x^{m-1}(ax+b)^n\,dx \\[2ex] \dfrac{-x^{m+1}(ax+b)^{n+1}}{(n+1)b} + \dfrac{m+n+2}{(n+1)b}\displaystyle\int x^m(ax+b)^{n+1}\,dx \end{cases}$$

## INTEGRALS INVOLVING $\sqrt{ax+b}$

**14.84** $$\int \frac{dx}{\sqrt{ax+b}} = \frac{2\sqrt{ax+b}}{a}$$

**14.85** $$\int \frac{x\,dx}{\sqrt{ax+b}} = \frac{2(ax-2b)}{3a^2}\sqrt{ax+b}$$

**14.86** $$\int \frac{x^2\,dx}{\sqrt{ax+b}} = \frac{2(3a^2x^2-4abx+8b^2)}{15a^3}\sqrt{ax+b}$$

**14.87** $$\int \frac{dx}{x\sqrt{ax+b}} = \begin{cases} \dfrac{1}{\sqrt{b}}\ln\left(\dfrac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right) \\[2ex] \dfrac{2}{\sqrt{-b}}\tan^{-1}\sqrt{\dfrac{ax+b}{-b}} \end{cases}$$

**14.88** $$\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{ax+b}}$$ [See 14.87]

**14.89** $$\int \sqrt{ax+b}\,dx = \frac{2\sqrt{(ax+b)^3}}{3a}$$

**14.90** $$\int x\sqrt{ax+b}\,dx = \frac{2(3ax-2b)}{15a^2}\sqrt{(ax+b)^3}$$

**14.91** $$\int x^2\sqrt{ax+b}\,dx = \frac{2(15a^2x^2-12abx+8b^2)}{105a^3}\sqrt{(ax+b)^3}$$

**14.92** $$\int \frac{\sqrt{ax+b}}{x}\,dx = 2\sqrt{ax+b} + b\int \frac{dx}{x\sqrt{ax+b}} \qquad \text{[See 14.87]}$$

**14.93** $$\int \frac{\sqrt{ax+b}}{x^2}\,dx = -\frac{\sqrt{ax+b}}{x} + \frac{a}{2}\int \frac{dx}{x\sqrt{ax+b}} \qquad \text{[See 14.87]}$$

**14.94** $$\int \frac{x^m}{\sqrt{ax+b}}\,dx = \frac{2x^m\sqrt{ax+b}}{(2m+1)a} - \frac{2mb}{(2m+1)a}\int \frac{x^{m-1}}{\sqrt{ax+b}}\,dx$$

**14.95** $$\int \frac{dx}{x^m\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{(m-1)bx^{m-1}} - \frac{(2m-3)a}{(2m-2)b}\int \frac{dx}{x^{m-1}\sqrt{ax+b}}$$

**14.96** $$\int x^m\sqrt{ax+b}\,dx = \frac{2x^m}{(2m+3)a}(ax+b)^{3/2} - \frac{2mb}{(2m+3)a}\int x^{m-1}\sqrt{ax+b}\,dx$$

**14.97** $$\int \frac{\sqrt{ax+b}}{x^m}\,dx = -\frac{\sqrt{ax+b}}{(m-1)x^{m-1}} + \frac{a}{2(m-1)}\int \frac{dx}{x^{m-1}\sqrt{ax+b}}$$

**14.98** $$\int \frac{\sqrt{ax+b}}{x^m}\,dx = \frac{-(ax+b)^{3/2}}{(m-1)bx^{m-1}} - \frac{(2m-5)a}{(2m-2)b}\int \frac{\sqrt{ax+b}}{x^{m-1}}\,dx$$

**14.99** $$\int (ax+b)^{m/2}\,dx = \frac{2(ax+b)^{(m+2)/2}}{a(m+2)}$$

**14.100** $$\int x(ax+b)^{m/2}\,dx = \frac{2(ax+b)^{(m+4)/2}}{a^2(m+4)} - \frac{2b(ax+b)^{(m+2)/2}}{a^2(m+2)}$$

**14.101** $$\int x^2(ax+b)^{m/2}\,dx = \frac{2(ax+b)^{(m+6)/2}}{a^3(m+6)} - \frac{4b(ax+b)^{(m+4)/2}}{a^3(m+4)} + \frac{2b^2(ax+b)^{(m+2)/2}}{a^3(m+2)}$$

**14.102** $$\int \frac{(ax+b)^{m/2}}{x}\,dx = \frac{2(ax+b)^{m/2}}{m} + b\int \frac{(ax+b)^{(m-2)/2}}{x}\,dx$$

**14.103** $$\int \frac{(ax+b)^{m/2}}{x^2}\,dx = -\frac{(ax+b)^{(m+2)/2}}{bx} + \frac{ma}{2b}\int \frac{(ax+b)^{m/2}}{x}\,dx$$

**14.104** $$\int \frac{dx}{x(ax+b)^{m/2}} = \frac{2}{(m-2)b(ax+b)^{(m-2)/2}} + \frac{1}{b}\int \frac{dx}{x(ax+b)^{(m-2)/2}}$$

## INTEGRALS INVOLVING $ax+b$ AND $px+q$

**14.105** $$\int \frac{dx}{(ax+b)(px+q)} = \frac{1}{bp-aq}\ln\left(\frac{px+q}{ax+b}\right)$$

**14.106** $$\int \frac{x\,dx}{(ax+b)(px+q)} = \frac{1}{bp-aq}\left\{\frac{b}{a}\ln(ax+b) - \frac{q}{p}\ln(px+q)\right\}$$

**14.107** $$\int \frac{dx}{(ax+b)^2(px+q)} = \frac{1}{bp-aq}\left\{\frac{1}{ax+b} + \frac{p}{bp-aq}\ln\left(\frac{px+q}{ax+b}\right)\right\}$$

**14.108** $$\int \frac{x\,dx}{(ax+b)^2(px+q)} = \frac{1}{bp-aq}\left\{\frac{q}{bp-aq}\ln\left(\frac{ax+b}{px+q}\right) - \frac{b}{a(ax+b)}\right\}$$

**14.109** $$\int \frac{x^2\,dx}{(ax+b)^2(px+q)} = \frac{b^2}{(bp-aq)a^2(ax+b)} + \frac{1}{(bp-aq)^2}\left\{\frac{q^2}{p}\ln(px+q) + \frac{b(bp-2aq)}{a^2}\ln(ax+b)\right\}$$

**14.110** $$\int \frac{dx}{(ax+b)^m(px+q)^n} = \frac{-1}{(n-1)(bp-aq)}\left\{\frac{1}{(ax+b)^{m-1}(px+q)^{n-1}} + a(m+n-2)\int\frac{dx}{(ax+b)^m(px+q)^{n-1}}\right\}$$

**14.111** $$\int \frac{ax+b}{px+q}\,dx = \frac{ax}{p} + \frac{bp-aq}{p^2}\ln(px+q)$$

**14.112** $$\int \frac{(ax+b)^m}{(px+q)^n}\,dx = \begin{cases} \dfrac{-1}{(n-1)(bp-aq)}\left\{\dfrac{(ax+b)^{m+1}}{(px+q)^{n-1}} + (n-m-2)a\displaystyle\int\frac{(ax+b)^m}{(px+q)^{n-1}}\,dx\right\} \\ \dfrac{-1}{(n-m-1)p}\left\{\dfrac{(ax+b)^m}{(px+q)^{n-1}} + m(bp-aq)\displaystyle\int\frac{(ax+b)^{m-1}}{(px+q)^n}\,dx\right\} \\ \dfrac{-1}{(n-1)p}\left\{\dfrac{(ax+b)^m}{(px+q)^{n-1}} - ma\displaystyle\int\frac{(ax+b)^{m-1}}{(px+q)^{n-1}}\,dx\right\} \end{cases}$$

## INTEGRALS INVOLVING $\sqrt{ax+b}$ AND $px+q$

**14.113** $$\int \frac{px+q}{\sqrt{ax+b}}\,dx = \frac{2(apx+3aq-2bp)}{3a^2}\sqrt{ax+b}$$

**14.114** $$\int \frac{dx}{(px+q)\sqrt{ax+b}} = \begin{cases} \dfrac{1}{\sqrt{bp-aq}\,\sqrt{p}}\ln\left(\dfrac{\sqrt{p(ax+b)}-\sqrt{bp-aq}}{\sqrt{p(ax+b)}+\sqrt{bp-aq}}\right) \\ \dfrac{2}{\sqrt{aq-bp}\,\sqrt{p}}\tan^{-1}\sqrt{\dfrac{p(ax+b)}{aq-bp}} \end{cases}$$

**14.115** $$\int \frac{\sqrt{ax+b}}{px+q}\,dx = \begin{cases} \dfrac{2\sqrt{ax+b}}{p} + \dfrac{\sqrt{bp-aq}}{p\sqrt{p}}\ln\left(\dfrac{\sqrt{p(ax+b)}-\sqrt{bp-aq}}{\sqrt{p(ax+b)}+\sqrt{bp-aq}}\right) \\ \dfrac{2\sqrt{ax+b}}{p} - \dfrac{2\sqrt{aq-bp}}{p\sqrt{p}}\tan^{-1}\sqrt{\dfrac{p(ax+b)}{aq-bp}} \end{cases}$$

**14.116** $$\int (px+q)^n\sqrt{ax+b}\,dx = \frac{2(px+q)^{n+1}\sqrt{ax+b}}{(2n+3)p} + \frac{bp-aq}{(2n+3)p}\int\frac{(px+q)^n}{\sqrt{ax+b}}\,dx$$

**14.117** $$\int \frac{dx}{(px+q)^n\sqrt{ax+b}} = \frac{\sqrt{ax+b}}{(n-1)(aq-bp)(px+q)^{n-1}} + \frac{(2n-3)a}{2(n-1)(aq-bp)}\int\frac{dx}{(px+q)^{n-1}\sqrt{ax+b}}$$

**14.118** $$\int \frac{(px+q)^n}{\sqrt{ax+b}}\,dx = \frac{2(px+q)^n\sqrt{ax+b}}{(2n+1)a} + \frac{2n(aq-bp)}{(2n+1)a}\int\frac{(px+q)^{n-1}\,dx}{\sqrt{ax+b}}$$

**14.119** $$\int \frac{\sqrt{ax+b}}{(px+q)^n}\,dx = \frac{-\sqrt{ax+b}}{(n-1)p(px+q)^{n-1}} + \frac{a}{2(n-1)p}\int\frac{dx}{(px+q)^{n-1}\sqrt{ax+b}}$$

## INTEGRALS INVOLVING $\sqrt{ax+b}$ AND $\sqrt{px+q}$

**14.120** $$\int \frac{dx}{\sqrt{(ax+b)(px+q)}} = \begin{cases} \dfrac{2}{\sqrt{ap}}\ln\left(\sqrt{a(px+q)}+\sqrt{p(ax+b)}\right) \\ \dfrac{2}{\sqrt{-ap}}\tan^{-1}\sqrt{\dfrac{-p(ax+b)}{a(px+q)}} \end{cases}$$

**14.121** $$\int \frac{x\,dx}{\sqrt{(ax+b)(px+q)}} = \frac{\sqrt{(ax+b)(px+q)}}{ap} - \frac{bp+aq}{2ap}\int\frac{dx}{\sqrt{(ax+b)(px+q)}}$$

**14.122** $$\int \sqrt{(ax+b)(px+q)}\,dx \;=\; \frac{2apx+bp+aq}{4ap}\sqrt{(ax+b)(px+q)} - \frac{(bp-aq)^2}{8ap}\int \frac{dx}{\sqrt{(ax+b)(px+q)}}$$

**14.123** $$\int \sqrt{\frac{px+q}{ax+b}}\,dx \;=\; \frac{\sqrt{(ax+b)(px+q)}}{a} + \frac{aq-bp}{2a}\int \frac{dx}{\sqrt{(ax+b)(px+q)}}$$

**14.124** $$\int \frac{dx}{(px+q)\sqrt{(ax+b)(px+q)}} \;=\; \frac{2\sqrt{ax+b}}{(aq-bp)\sqrt{px+q}}$$

## INTEGRALS INVOLVING $x^2 + a^2$

**14.125** $$\int \frac{dx}{x^2+a^2} \;=\; \frac{1}{a}\tan^{-1}\frac{x}{a}$$

**14.126** $$\int \frac{x\,dx}{x^2+a^2} \;=\; \frac{1}{2}\ln(x^2+a^2)$$

**14.127** $$\int \frac{x^2\,dx}{x^2+a^2} \;=\; x - a\tan^{-1}\frac{x}{a}$$

**14.128** $$\int \frac{x^3\,dx}{x^2+a^2} \;=\; \frac{x^2}{2} - \frac{a^2}{2}\ln(x^2+a^2)$$

**14.129** $$\int \frac{dx}{x(x^2+a^2)} \;=\; \frac{1}{2a^2}\ln\left(\frac{x^2}{x^2+a^2}\right)$$

**14.130** $$\int \frac{dx}{x^2(x^2+a^2)} \;=\; -\frac{1}{a^2x} - \frac{1}{a^3}\tan^{-1}\frac{x}{a}$$

**14.131** $$\int \frac{dx}{x^3(x^2+a^2)} \;=\; -\frac{1}{2a^2x^2} - \frac{1}{2a^4}\ln\left(\frac{x^2}{x^2+a^2}\right)$$

**14.132** $$\int \frac{dx}{(x^2+a^2)^2} \;=\; \frac{x}{2a^2(x^2+a^2)} + \frac{1}{2a^3}\tan^{-1}\frac{x}{a}$$

**14.133** $$\int \frac{x\,dx}{(x^2+a^2)^2} \;=\; \frac{-1}{2(x^2+a^2)}$$

**14.134** $$\int \frac{x^2\,dx}{(x^2+a^2)^2} \;=\; \frac{-x}{2(x^2+a^2)} + \frac{1}{2a}\tan^{-1}\frac{x}{a}$$

**14.135** $$\int \frac{x^3\,dx}{(x^2+a^2)^2} \;=\; \frac{a^2}{2(x^2+a^2)} + \frac{1}{2}\ln(x^2+a^2)$$

**14.136** $$\int \frac{dx}{x(x^2+a^2)^2} \;=\; \frac{1}{2a^2(x^2+a^2)} + \frac{1}{2a^4}\ln\left(\frac{x^2}{x^2+a^2}\right)$$

**14.137** $$\int \frac{dx}{x^2(x^2+a^2)^2} \;=\; -\frac{1}{a^4x} - \frac{x}{2a^4(x^2+a^2)} - \frac{3}{2a^5}\tan^{-1}\frac{x}{a}$$

**14.138** $$\int \frac{dx}{x^3(x^2+a^2)^2} \;=\; -\frac{1}{2a^4x^2} - \frac{1}{2a^4(x^2+a^2)} - \frac{1}{a^6}\ln\left(\frac{x^2}{x^2+a^2}\right)$$

**14.139** $$\int \frac{dx}{(x^2+a^2)^n} \;=\; \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{2n-3}{(2n-2)a^2}\int \frac{dx}{(x^2+a^2)^{n-1}}$$

**14.140** $$\int \frac{x\,dx}{(x^2+a^2)^n} \;=\; \frac{-1}{2(n-1)(x^2+a^2)^{n-1}}$$

**14.141** $$\int \frac{dx}{x(x^2+a^2)^n} \;=\; \frac{1}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{1}{a^2}\int \frac{dx}{x(x^2+a^2)^{n-1}}$$

**14.142** $$\int \frac{x^m\,dx}{(x^2+a^2)^n} \;=\; \int \frac{x^{m-2}\,dx}{(x^2+a^2)^{n-1}} - a^2\int \frac{x^{m-2}\,dx}{(x^2+a^2)^n}$$

**14.143** $$\int \frac{dx}{x^m(x^2+a^2)^n} \;=\; \frac{1}{a^2}\int \frac{dx}{x^m(x^2+a^2)^{n-1}} - \frac{1}{a^2}\int \frac{dx}{x^{m-2}(x^2+a^2)^n}$$

## INTEGRALS INVOLVING $x^2 - a^2, \quad x^2 > a^2$

**14.144** $\displaystyle\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) \quad \text{or} \quad -\frac{1}{a}\coth^{-1}\frac{x}{a}$

**14.145** $\displaystyle\int \frac{x\,dx}{x^2 - a^2} = \frac{1}{2}\ln(x^2 - a^2)$

**14.146** $\displaystyle\int \frac{x^2\,dx}{x^2 - a^2} = x + \frac{a}{2}\ln\left(\frac{x-a}{x+a}\right)$

**14.147** $\displaystyle\int \frac{x^3\,dx}{x^2 - a^2} = \frac{x^2}{2} + \frac{a^2}{2}\ln(x^2 - a^2)$

**14.148** $\displaystyle\int \frac{dx}{x(x^2 - a^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2 - a^2}{x^2}\right)$

**14.149** $\displaystyle\int \frac{dx}{x^2(x^2 - a^2)} = \frac{1}{a^2 x} + \frac{1}{2a^3}\ln\left(\frac{x-a}{x+a}\right)$

**14.150** $\displaystyle\int \frac{dx}{x^3(x^2 - a^2)} = \frac{1}{2a^2x^2} - \frac{1}{2a^4}\ln\left(\frac{x^2}{x^2 - a^2}\right)$

**14.151** $\displaystyle\int \frac{dx}{(x^2 - a^2)^2} = \frac{-x}{2a^2(x^2 - a^2)} - \frac{1}{4a^3}\ln\left(\frac{x-a}{x+a}\right)$

**14.152** $\displaystyle\int \frac{x\,dx}{(x^2 - a^2)^2} = \frac{-1}{2(x^2 - a^2)}$

**14.153** $\displaystyle\int \frac{x^2\,dx}{(x^2 - a^2)^2} = \frac{-x}{2(x^2 - a^2)} + \frac{1}{4a}\ln\left(\frac{x-a}{x+a}\right)$

**14.154** $\displaystyle\int \frac{x^3\,dx}{(x^2 - a^2)^2} = \frac{-a^2}{2(x^2 - a^2)} + \frac{1}{2}\ln(x^2 - a^2)$

**14.155** $\displaystyle\int \frac{dx}{x(x^2 - a^2)^2} = \frac{-1}{2a^2(x^2 - a^2)} + \frac{1}{2a^4}\ln\left(\frac{x^2}{x^2 - a^2}\right)$

**14.156** $\displaystyle\int \frac{dx}{x^2(x^2 - a^2)^2} = -\frac{1}{a^4 x} - \frac{x}{2a^4(x^2 - a^2)} - \frac{3}{4a^5}\ln\left(\frac{x-a}{x+a}\right)$

**14.157** $\displaystyle\int \frac{dx}{x^3(x^2 - a^2)^2} = -\frac{1}{2a^4x^2} - \frac{1}{2a^4(x^2 - a^2)} + \frac{1}{a^6}\ln\left(\frac{x^2}{x^2 - a^2}\right)$

**14.158** $\displaystyle\int \frac{dx}{(x^2 - a^2)^n} = \frac{-x}{2(n-1)a^2(x^2 - a^2)^{n-1}} - \frac{2n-3}{(2n-2)a^2}\int \frac{dx}{(x^2 - a^2)^{n-1}}$

**14.159** $\displaystyle\int \frac{x\,dx}{(x^2 - a^2)^n} = \frac{-1}{2(n-1)(x^2 - a^2)^{n-1}}$

**14.160** $\displaystyle\int \frac{dx}{x(x^2 - a^2)^n} = \frac{-1}{2(n-1)a^2(x^2 - a^2)^{n-1}} - \frac{1}{a^2}\int \frac{dx}{x(x^2 - a^2)^{n-1}}$

**14.161** $\displaystyle\int \frac{x^m\,dx}{(x^2 - a^2)^n} = \int \frac{x^{m-2}\,dx}{(x^2 - a^2)^{n-1}} + a^2\int \frac{x^{m-2}\,dx}{(x^2 - a^2)^n}$

**14.162** $\displaystyle\int \frac{dx}{x^m(x^2 - a^2)^n} = \frac{1}{a^2}\int \frac{dx}{x^{m-2}(x^2 - a^2)^n} - \frac{1}{a^2}\int \frac{dx}{x^m(x^2 - a^2)^{n-1}}$

## INTEGRALS INVOLVING $a^2 - x^2$, $x^2 < a^2$

**14.163** $$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln\left(\frac{a+x}{a-x}\right) \quad \text{or} \quad \frac{1}{a} \tanh^{-1} \frac{x}{a}$$

**14.164** $$\int \frac{x\,dx}{a^2 - x^2} = -\frac{1}{2} \ln(a^2 - x^2)$$

**14.165** $$\int \frac{x^2\,dx}{a^2 - x^2} = -x + \frac{a}{2} \ln\left(\frac{a+x}{a-x}\right)$$

**14.166** $$\int \frac{x^3\,dx}{a^2 - x^2} = -\frac{x^2}{2} - \frac{a^2}{2} \ln(a^2 - x^2)$$

**14.167** $$\int \frac{dx}{x(a^2 - x^2)} = \frac{1}{2a^2} \ln\left(\frac{x^2}{a^2 - x^2}\right)$$

**14.168** $$\int \frac{dx}{x^2(a^2 - x^2)} = -\frac{1}{a^2 x} + \frac{1}{2a^3} \ln\left(\frac{a+x}{a-x}\right)$$

**14.169** $$\int \frac{dx}{x^3(a^2 - x^2)} = -\frac{1}{2a^2 x^2} + \frac{1}{2a^4} \ln\left(\frac{x^2}{a^2 - x^2}\right)$$

**14.170** $$\int \frac{dx}{(a^2 - x^2)^2} = \frac{x}{2a^2(a^2 - x^2)} + \frac{1}{4a^3} \ln\left(\frac{a+x}{a-x}\right)$$

**14.171** $$\int \frac{x\,dx}{(a^2 - x^2)^2} = \frac{1}{2(a^2 - x^2)}$$

**14.172** $$\int \frac{x^2\,dx}{(a^2 - x^2)^2} = \frac{x}{2(a^2 - x^2)} - \frac{1}{4a} \ln\left(\frac{a+x}{a-x}\right)$$

**14.173** $$\int \frac{x^3\,dx}{(a^2 - x^2)^2} = \frac{a^2}{2(a^2 - x^2)} + \frac{1}{2} \ln(a^2 - x^2)$$

**14.174** $$\int \frac{dx}{x(a^2 - x^2)^2} = \frac{1}{2a^2(a^2 - x^2)} + \frac{1}{2a^4} \ln\left(\frac{x^2}{a^2 - x^2}\right)$$

**14.175** $$\int \frac{dx}{x^2(a^2 - x^2)^2} = \frac{-1}{a^4 x} + \frac{x}{2a^4(a^2 - x^2)} + \frac{3}{4a^5} \ln\left(\frac{a+x}{a-x}\right)$$

**14.176** $$\int \frac{dx}{x^3(a^2 - x^2)^2} = \frac{-1}{2a^4 x^2} + \frac{1}{2a^4(a^2 - x^2)} + \frac{1}{a^6} \ln\left(\frac{x^2}{a^2 - x^2}\right)$$

**14.177** $$\int \frac{dx}{(a^2 - x^2)^n} = \frac{x}{2(n-1)a^2(a^2 - x^2)^{n-1}} + \frac{2n-3}{(2n-2)a^2} \int \frac{dx}{(a^2 - x^2)^{n-1}}$$

**14.178** $$\int \frac{x\,dx}{(a^2 - x^2)^n} = \frac{1}{2(n-1)(a^2 - x^2)^{n-1}}$$

**14.179** $$\int \frac{dx}{x(a^2 - x^2)^n} = \frac{1}{2(n-1)a^2(a^2 - x^2)^{n-1}} + \frac{1}{a^2} \int \frac{dx}{x(a^2 - x^2)^{n-1}}$$

**14.180** $$\int \frac{x^m\,dx}{(a^2 - x^2)^n} = a^2 \int \frac{x^{m-2}dx}{(a^2 - x^2)^n} - \int \frac{x^{m-2}dx}{(a^2 - x^2)^{n-1}}$$

**14.181** $$\int \frac{dx}{x^m(a^2 - x^2)^n} = \frac{1}{a^2} \int \frac{dx}{x^m(a^2 - x^2)^{n-1}} + \frac{1}{a^2} \int \frac{dx}{x^{m-2}(a^2 - x^2)^n}$$

## INTEGRALS INVOLVING $\sqrt{x^2+a^2}$

**14.182** $\displaystyle \int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x+\sqrt{x^2+a^2}\right) \quad \text{or} \quad \sinh^{-1}\frac{x}{a}$

**14.183** $\displaystyle \int \frac{x\,dx}{\sqrt{x^2+a^2}} = \sqrt{x^2+a^2}$

**14.184** $\displaystyle \int \frac{x^2\,dx}{\sqrt{x^2+a^2}} = \frac{x\sqrt{x^2+a^2}}{2} - \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right)$

**14.185** $\displaystyle \int \frac{x^3\,dx}{\sqrt{x^2+a^2}} = \frac{(x^2+a^2)^{3/2}}{3} - a^2\sqrt{x^2+a^2}$

**14.186** $\displaystyle \int \frac{dx}{x\sqrt{x^2+a^2}} = -\frac{1}{a}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$

**14.187** $\displaystyle \int \frac{dx}{x^2\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{a^2x}$

**14.188** $\displaystyle \int \frac{dx}{x^3\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{2a^2x^2} + \frac{1}{2a^3}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$

**14.189** $\displaystyle \int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right)$

**14.190** $\displaystyle \int x\sqrt{x^2+a^2}\,dx = \frac{(x^2+a^2)^{3/2}}{3}$

**14.191** $\displaystyle \int x^2\sqrt{x^2+a^2}\,dx = \frac{x(x^2+a^2)^{3/2}}{4} - \frac{a^2x\sqrt{x^2+a^2}}{8} - \frac{a^4}{8}\ln\left(x+\sqrt{x^2+a^2}\right)$

**14.192** $\displaystyle \int x^3\sqrt{x^2+a^2}\,dx = \frac{(x^2+a^2)^{5/2}}{5} - \frac{a^2(x^2+a^2)^{3/2}}{3}$

**14.193** $\displaystyle \int \frac{\sqrt{x^2+a^2}}{x}\,dx = \sqrt{x^2+a^2} - a\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$

**14.194** $\displaystyle \int \frac{\sqrt{x^2+a^2}}{x^2}\,dx = -\frac{\sqrt{x^2+a^2}}{x} + \ln\left(x+\sqrt{x^2+a^2}\right)$

**14.195** $\displaystyle \int \frac{\sqrt{x^2+a^2}}{x^3}\,dx = -\frac{\sqrt{x^2+a^2}}{2x^2} - \frac{1}{2a}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$

**14.196** $\displaystyle \int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}}$

**14.197** $\displaystyle \int \frac{x\,dx}{(x^2+a^2)^{3/2}} = \frac{-1}{\sqrt{x^2+a^2}}$

**14.198** $\displaystyle \int \frac{x^2\,dx}{(x^2+a^2)^{3/2}} = \frac{-x}{\sqrt{x^2+a^2}} + \ln\left(x+\sqrt{x^2+a^2}\right)$

**14.199** $\displaystyle \int \frac{x^3\,dx}{(x^2+a^2)^{3/2}} = \sqrt{x^2+a^2} + \frac{a^2}{\sqrt{x^2+a^2}}$

**14.200** $\displaystyle \int \frac{dx}{x(x^2+a^2)^{3/2}} = \frac{1}{a^2\sqrt{x^2+a^2}} - \frac{1}{a^3}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$

**14.201** $\displaystyle \int \frac{dx}{x^2(x^2+a^2)^{3/2}} = -\frac{\sqrt{x^2+a^2}}{a^4x} - \frac{x}{a^4\sqrt{x^2+a^2}}$

**14.202** $\displaystyle \int \frac{dx}{x^3(x^2+a^2)^{3/2}} = \frac{-1}{2a^2x^2\sqrt{x^2+a^2}} - \frac{3}{2a^4\sqrt{x^2+a^2}} + \frac{3}{2a^5}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$

**14.203** $\displaystyle\int (x^2+a^2)^{3/2}\,dx = \frac{x(x^2+a^2)^{3/2}}{4} + \frac{3a^2x\sqrt{x^2+a^2}}{8} + \frac{3}{8}a^4\ln\left(x+\sqrt{x^2+a^2}\right)$

**14.204** $\displaystyle\int x(x^2+a^2)^{3/2}\,dx = \frac{(x^2+a^2)^{5/2}}{5}$

**14.205** $\displaystyle\int x^2(x^2+a^2)^{3/2}\,dx = \frac{x(x^2+a^2)^{5/2}}{6} - \frac{a^2x(x^2+a^2)^{3/2}}{24} - \frac{a^4x\sqrt{x^2+a^2}}{16} - \frac{a^6}{16}\ln\left(x+\sqrt{x^2+a^2}\right)$

**14.206** $\displaystyle\int x^3(x^2+a^2)^{3/2}\,dx = \frac{(x^2+a^2)^{7/2}}{7} - \frac{a^2(x^2+a^2)^{5/2}}{5}$

**14.207** $\displaystyle\int \frac{(x^2+a^2)^{3/2}}{x}\,dx = \frac{(x^2+a^2)^{3/2}}{3} + a^2\sqrt{x^2+a^2} - a^3\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$

**14.208** $\displaystyle\int \frac{(x^2+a^2)^{3/2}}{x^2}\,dx = -\frac{(x^2+a^2)^{3/2}}{x} + \frac{3x\sqrt{x^2+a^2}}{2} + \frac{3}{2}a^2\ln\left(x+\sqrt{x^2+a^2}\right)$

**14.209** $\displaystyle\int \frac{(x^2+a^2)^{3/2}}{x^3}\,dx = -\frac{(x^2+a^2)^{3/2}}{2x^2} + \frac{3}{2}\sqrt{x^2+a^2} - \frac{3}{2}a\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$

## INTEGRALS INVOLVING $\sqrt{x^2-a^2}$

**14.210** $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left(x+\sqrt{x^2-a^2}\right), \qquad \int \frac{x\,dx}{\sqrt{x^2-a^2}} = \sqrt{x^2-a^2}$

**14.211** $\displaystyle\int \frac{x^2\,dx}{\sqrt{x^2-a^2}} = \frac{x\sqrt{x^2-a^2}}{2} + \frac{a^2}{2}\ln\left(x+\sqrt{x^2-a^2}\right)$

**14.212** $\displaystyle\int \frac{x^3\,dx}{\sqrt{x^2-a^2}} = \frac{(x^2-a^2)^{3/2}}{3} + a^2\sqrt{x^2-a^2}$

**14.213** $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right|$

**14.214** $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2x}$

**14.215** $\displaystyle\int \frac{dx}{x^3\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{2a^2x^2} + \frac{1}{2a^3}\sec^{-1}\left|\frac{x}{a}\right|$

**14.216** $\displaystyle\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\ln\left(x+\sqrt{x^2-a^2}\right)$

**14.217** $\displaystyle\int x\sqrt{x^2-a^2}\,dx = \frac{(x^2-a^2)^{3/2}}{3}$

**14.218** $\displaystyle\int x^2\sqrt{x^2-a^2}\,dx = \frac{x(x^2-a^2)^{3/2}}{4} + \frac{a^2x\sqrt{x^2-a^2}}{8} - \frac{a^4}{8}\ln\left(x+\sqrt{x^2-a^2}\right)$

**14.219** $\displaystyle\int x^3\sqrt{x^2-a^2}\,dx = \frac{(x^2-a^2)^{5/2}}{5} + \frac{a^2(x^2-a^2)^{3/2}}{3}$

**14.220** $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x}\,dx = \sqrt{x^2-a^2} - a\sec^{-1}\left|\frac{x}{a}\right|$

**14.221** $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x^2}\,dx = -\frac{\sqrt{x^2-a^2}}{x} + \ln\left(x+\sqrt{x^2-a^2}\right)$

**14.222** $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x^3}\,dx = -\frac{\sqrt{x^2-a^2}}{2x^2} + \frac{1}{2a}\sec^{-1}\left|\frac{x}{a}\right|$

**14.223** $\displaystyle\int \frac{dx}{(x^2-a^2)^{3/2}} = -\frac{x}{a^2\sqrt{x^2-a^2}}$

**14.224** $\displaystyle\int \frac{x\,dx}{(x^2-a^2)^{3/2}} = \frac{-1}{\sqrt{x^2-a^2}}$

**14.225** $\displaystyle\int \frac{x^2\,dx}{(x^2-a^2)^{3/2}} = -\frac{x}{\sqrt{x^2-a^2}} + \ln\,(x+\sqrt{x^2-a^2}\,)$

**14.226** $\displaystyle\int \frac{x^3\,dx}{(x^2-a^2)^{3/2}} = \sqrt{x^2-a^2} - \frac{a^2}{\sqrt{x^2-a^2}}$

**14.227** $\displaystyle\int \frac{dx}{x(x^2-a^2)^{3/2}} = \frac{-1}{a^2\sqrt{x^2-a^2}} - \frac{1}{a^3}\sec^{-1}\left|\frac{x}{a}\right|$

**14.228** $\displaystyle\int \frac{dx}{x^2(x^2-a^2)^{3/2}} = -\frac{\sqrt{x^2-a^2}}{a^4x} - \frac{x}{a^4\sqrt{x^2-a^2}}$

**14.229** $\displaystyle\int \frac{dx}{x^3(x^2-a^2)^{3/2}} = \frac{1}{2a^2x^2\sqrt{x^2-a^2}} - \frac{3}{2a^4\sqrt{x^2-a^2}} - \frac{3}{2a^5}\sec^{-1}\left|\frac{x}{a}\right|$

**14.230** $\displaystyle\int (x^2-a^2)^{3/2}\,dx = \frac{x(x^2-a^2)^{3/2}}{4} - \frac{3a^2x\sqrt{x^2-a^2}}{8} + \frac{3}{8}a^4\ln\,(x+\sqrt{x^2-a^2}\,)$

**14.231** $\displaystyle\int x(x^2-a^2)^{3/2}\,dx = \frac{(x^2-a^2)^{5/2}}{5}$

**14.232** $\displaystyle\int x^2(x^2-a^2)^{3/2}\,dx = \frac{x(x^2-a^2)^{5/2}}{6} + \frac{a^2x(x^2-a^2)^{3/2}}{24} - \frac{a^4x\sqrt{x^2-a^2}}{16} + \frac{a^6}{16}\ln\,(x+\sqrt{x^2-a^2}\,)$

**14.233** $\displaystyle\int x^3(x^2-a^2)^{3/2}\,dx = \frac{(x^2-a^2)^{7/2}}{7} + \frac{a^2(x^2-a^2)^{5/2}}{5}$

**14.234** $\displaystyle\int \frac{(x^2-a^2)^{3/2}}{x}\,dx = \frac{(x^2-a^2)^{3/2}}{3} - a^2\sqrt{x^2-a^2} + a^3\sec^{-1}\left|\frac{x}{a}\right|$

**14.235** $\displaystyle\int \frac{(x^2-a^2)^{3/2}}{x^2}\,dx = -\frac{(x^2-a^2)^{3/2}}{x} + \frac{3x\sqrt{x^2-a^2}}{2} - \frac{3}{2}a^2\ln\,(x+\sqrt{x^2-a^2}\,)$

**14.236** $\displaystyle\int \frac{(x^2-a^2)^{3/2}}{x^3}\,dx = -\frac{(x^2-a^2)^{3/2}}{2x^2} + \frac{3\sqrt{x^2-a^2}}{2} - \frac{3}{2}a\sec^{-1}\left|\frac{x}{a}\right|$

## INTEGRALS INVOLVING $\sqrt{a^2-x^2}$

**14.237** $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$

**14.238** $\displaystyle\int \frac{x\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}$

**14.239** $\displaystyle\int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = -\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

**14.240** $\displaystyle\int \frac{x^3\,dx}{\sqrt{a^2-x^2}} = \frac{(a^2-x^2)^{3/2}}{3} - a^2\sqrt{a^2-x^2}$

**14.241** $\displaystyle\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$

**14.242** $\displaystyle\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x}$

**14.243** $\displaystyle\int \frac{dx}{x^3\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{2a^2x^2} - \frac{1}{2a^3}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$

**14.244** $$\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

**14.245** $$\int x\sqrt{a^2-x^2}\,dx = -\frac{(a^2-x^2)^{3/2}}{3}$$

**14.246** $$\int x^2\sqrt{a^2-x^2}\,dx = -\frac{x(a^2-x^2)^{3/2}}{4} + \frac{a^2x\sqrt{a^2-x^2}}{8} + \frac{a^4}{8}\sin^{-1}\frac{x}{a}$$

**14.247** $$\int x^3\sqrt{a^2-x^2}\,dx = \frac{(a^2-x^2)^{5/2}}{5} - \frac{a^2(a^2-x^2)^{3/2}}{3}$$

**14.248** $$\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

**14.249** $$\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx = -\frac{\sqrt{a^2-x^2}}{x} - \sin^{-1}\frac{x}{a}$$

**14.250** $$\int \frac{\sqrt{a^2-x^2}}{x^3}\,dx = -\frac{\sqrt{a^2-x^2}}{2x^2} + \frac{1}{2a}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

**14.251** $$\int \frac{dx}{(a^2-x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2-x^2}}$$

**14.252** $$\int \frac{x\,dx}{(a^2-x^2)^{3/2}} = \frac{1}{\sqrt{a^2-x^2}}$$

**14.253** $$\int \frac{x^2\,dx}{(a^2-x^2)^{3/2}} = \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{a}$$

**14.254** $$\int \frac{x^3\,dx}{(a^2-x^2)^{3/2}} = \sqrt{a^2-x^2} + \frac{a^2}{\sqrt{a^2-x^2}}$$

**14.255** $$\int \frac{dx}{x(a^2-x^2)^{3/2}} = \frac{1}{a^2\sqrt{a^2-x^2}} - \frac{1}{a^3}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

**14.256** $$\int \frac{dx}{x^2(a^2-x^2)^{3/2}} = -\frac{\sqrt{a^2-x^2}}{a^4x} + \frac{x}{a^4\sqrt{a^2-x^2}}$$

**14.257** $$\int \frac{dx}{x^3(a^2-x^2)^{3/2}} = \frac{-1}{2a^2x^2\sqrt{a^2-x^2}} + \frac{3}{2a^4\sqrt{a^2-x^2}} - \frac{3}{2a^5}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

**14.258** $$\int (a^2-x^2)^{3/2}\,dx = \frac{x(a^2-x^2)^{3/2}}{4} + \frac{3a^2x\sqrt{a^2-x^2}}{8} + \frac{3}{8}a^4\sin^{-1}\frac{x}{a}$$

**14.259** $$\int x(a^2-x^2)^{3/2}\,dx = -\frac{(a^2-x^2)^{5/2}}{5}$$

**14.260** $$\int x^2(a^2-x^2)^{3/2}\,dx = -\frac{x(a^2-x^2)^{5/2}}{6} + \frac{a^2x(a^2-x^2)^{3/2}}{24} + \frac{a^4x\sqrt{a^2-x^2}}{16} + \frac{a^6}{16}\sin^{-1}\frac{x}{a}$$

**14.261** $$\int x^3(a^2-x^2)^{3/2}\,dx = \frac{(a^2-x^2)^{7/2}}{7} - \frac{a^2(a^2-x^2)^{5/2}}{5}$$

**14.262** $$\int \frac{(a^2-x^2)^{3/2}}{x}\,dx = \frac{(a^2-x^2)^{3/2}}{3} + a^2\sqrt{a^2-x^2} - a^3\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

**14.263** $$\int \frac{(a^2-x^2)^{3/2}}{x^2}\,dx = -\frac{(a^2-x^2)^{3/2}}{x} - \frac{3x\sqrt{a^2-x^2}}{2} - \frac{3}{2}a^2\sin^{-1}\frac{x}{a}$$

**14.264** $$\int \frac{(a^2-x^2)^{3/2}}{x^3}\,dx = -\frac{(a^2-x^2)^{3/2}}{2x^2} - \frac{3\sqrt{a^2-x^2}}{2} + \frac{3}{2}a\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

## INTEGRALS INVOLVING $ax^2 + bx + c$

**14.265**
$$\int \frac{dx}{ax^2+bx+c} = \begin{cases} \dfrac{2}{\sqrt{4ac-b^2}} \tan^{-1} \dfrac{2ax+b}{\sqrt{4ac-b^2}} \\[2ex] \dfrac{1}{\sqrt{b^2-4ac}} \ln \left( \dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}} \right) \end{cases}$$

If $b^2 = 4ac$, $ax^2+bx+c = a(x+b/2a)^2$ and the results on pages 60-61 can be used. If $b = 0$ use results on page 64. If $a$ or $c = 0$ use results on pages 60-61.

**14.266**
$$\int \frac{x\,dx}{ax^2+bx+c} = \frac{1}{2a} \ln(ax^2+bx+c) - \frac{b}{2a} \int \frac{dx}{ax^2+bx+c}$$

**14.267**
$$\int \frac{x^2\,dx}{ax^2+bx+c} = \frac{x}{a} - \frac{b}{2a^2} \ln(ax^2+bx+c) + \frac{b^2-2ac}{2a^2} \int \frac{dx}{ax^2+bx+c}$$

**14.268**
$$\int \frac{x^m\,dx}{ax^2+bx+c} = \frac{x^{m-1}}{(m-1)a} - \frac{c}{a} \int \frac{x^{m-2}\,dx}{ax^2+bx+c} - \frac{b}{a} \int \frac{x^{m-1}\,dx}{ax^2+bx+c}$$

**14.269**
$$\int \frac{dx}{x(ax^2+bx+c)} = \frac{1}{2c} \ln \left( \frac{x^2}{ax^2+bx+c} \right) - \frac{b}{2c} \int \frac{dx}{ax^2+bx+c}$$

**14.270**
$$\int \frac{dx}{x^2(ax^2+bx+c)} = \frac{b}{2c^2} \ln \left( \frac{ax^2+bx+c}{x^2} \right) - \frac{1}{cx} + \frac{b^2-2ac}{2c^2} \int \frac{dx}{ax^2+bx+c}$$

**14.271**
$$\int \frac{dx}{x^n(ax^2+bx+c)} = -\frac{1}{(n-1)cx^{n-1}} - \frac{b}{c} \int \frac{dx}{x^{n-1}(ax^2+bx+c)} - \frac{a}{c} \int \frac{dx}{x^{n-2}(ax^2+bx+c)}$$

**14.272**
$$\int \frac{dx}{(ax^2+bx+c)^2} = \frac{2ax+b}{(4ac-b^2)(ax^2+bx+c)} + \frac{2a}{4ac-b^2} \int \frac{dx}{ax^2+bx+c}$$

**14.273**
$$\int \frac{x\,dx}{(ax^2+bx+c)^2} = -\frac{bx+2c}{(4ac-b^2)(ax^2+bx+c)} - \frac{b}{4ac-b^2} \int \frac{dx}{ax^2+bx+c}$$

**14.274**
$$\int \frac{x^2\,dx}{(ax^2+bx+c)^2} = \frac{(b^2-2ac)x+bc}{a(4ac-b^2)(ax^2+bx+c)} + \frac{2c}{4ac-b^2} \int \frac{dx}{ax^2+bx+c}$$

**14.275**
$$\int \frac{x^m\,dx}{(ax^2+bx+c)^n} = -\frac{x^{m-1}}{(2n-m-1)a(ax^2+bx+c)^{n-1}} + \frac{(m-1)c}{(2n-m-1)a} \int \frac{x^{m-2}\,dx}{(ax^2+bx+c)^n}$$
$$- \frac{(n-m)b}{(2n-m-1)a} \int \frac{x^{m-1}\,dx}{(ax^2+bx+c)^n}$$

**14.276**
$$\int \frac{x^{2n-1}\,dx}{(ax^2+bx+c)^n} = \frac{1}{a} \int \frac{x^{2n-3}\,dx}{(ax^2+bx+c)^{n-1}} - \frac{c}{a} \int \frac{x^{2n-3}\,dx}{(ax^2+bx+c)^n} - \frac{b}{a} \int \frac{x^{2n-2}\,dx}{(ax^2+bx+c)^n}$$

**14.277**
$$\int \frac{dx}{x(ax^2+bx+c)^2} = \frac{1}{2c(ax^2+bx+c)} - \frac{b}{2c} \int \frac{dx}{(ax^2+bx+c)^2} + \frac{1}{c} \int \frac{dx}{x(ax^2+bx+c)}$$

**14.278**
$$\int \frac{dx}{x^2(ax^2+bx+c)^2} = -\frac{1}{cx(ax^2+bx+c)} - \frac{3a}{c} \int \frac{dx}{(ax^2+bx+c)^2} - \frac{2b}{c} \int \frac{dx}{x(ax^2+bx+c)^2}$$

**14.279**
$$\int \frac{dx}{x^m(ax^2+bx+c)^n} = -\frac{1}{(m-1)cx^{m-1}(ax^2+bx+c)^{n-1}} - \frac{(m+2n-3)a}{(m-1)c} \int \frac{dx}{x^{m-2}(ax^2+bx+c)^n}$$
$$- \frac{(m+n-2)b}{(m-1)c} \int \frac{dx}{x^{m-1}(ax^2+bx+c)^n}$$

## INTEGRALS INVOLVING $\sqrt{ax^2+bx+c}$

**In the following results if $b^2 = 4ac$, $\sqrt{ax^2+bx+c} = \sqrt{a}\,(x + b/2a)$ and the results on pages 60-61 can be used. If $b = 0$ use the results on pages 67-70. If $a = 0$ or $c = 0$ use the results on pages 61-62.**

**14.280** $$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{1}{\sqrt{a}} \ln\left(2\sqrt{a}\,\sqrt{ax^2+bx+c} + 2ax + b\right) \\ -\dfrac{1}{\sqrt{-a}} \sin^{-1}\left(\dfrac{2ax+b}{\sqrt{b^2-4ac}}\right) \quad \text{or} \quad \dfrac{1}{\sqrt{a}} \sinh^{-1}\left(\dfrac{2ax+b}{\sqrt{4ac-b^2}}\right) \end{cases}$$

**14.281** $$\int \frac{x\,dx}{\sqrt{ax^2+bx+c}} = \frac{\sqrt{ax^2+bx+c}}{a} - \frac{b}{2a}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

**14.282** $$\int \frac{x^2\,dx}{\sqrt{ax^2+bx+c}} = \frac{2ax-3b}{4a^2}\sqrt{ax^2+bx+c} + \frac{3b^2-4ac}{8a^2}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

**14.283** $$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \begin{cases} -\dfrac{1}{\sqrt{c}} \ln\left(\dfrac{2\sqrt{c}\,\sqrt{ax^2+bx+c} + bx + 2c}{x}\right) \\ \dfrac{1}{\sqrt{-c}} \sin^{-1}\left(\dfrac{bx+2c}{|x|\,\sqrt{b^2-4ac}}\right) \quad \text{or} \quad -\dfrac{1}{\sqrt{c}} \sinh^{-1}\left(\dfrac{bx+2c}{|x|\,\sqrt{4ac-b^2}}\right) \end{cases}$$

**14.284** $$\int \frac{dx}{x^2\sqrt{ax^2+bx+c}} = -\frac{\sqrt{ax^2+bx+c}}{cx} - \frac{b}{2c}\int \frac{dx}{x\sqrt{ax^2+bx+c}}$$

**14.285** $$\int \sqrt{ax^2+bx+c}\,dx = \frac{(2ax+b)\sqrt{ax^2+bx+c}}{4a} + \frac{4ac-b^2}{8a}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

**14.286** $$\int x\sqrt{ax^2+bx+c}\,dx = \frac{(ax^2+bx+c)^{3/2}}{3a} - \frac{b(2ax+b)}{8a^2}\sqrt{ax^2+bx+c} - \frac{b(4ac-b^2)}{16a^2}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

**14.287** $$\int x^2\sqrt{ax^2+bx+c}\,dx = \frac{6ax-5b}{24a^2}(ax^2+bx+c)^{3/2} + \frac{5b^2-4ac}{16a^2}\int \sqrt{ax^2+bx+c}\,dx$$

**14.288** $$\int \frac{\sqrt{ax^2+bx+c}}{x}\,dx = \sqrt{ax^2+bx+c} + \frac{b}{2}\int \frac{dx}{\sqrt{ax^2+bx+c}} + c\int \frac{dx}{x\sqrt{ax^2+bx+c}}$$

**14.289** $$\int \frac{\sqrt{ax^2+bx+c}}{x^2}\,dx = -\frac{\sqrt{ax^2+bx+c}}{x} + a\int \frac{dx}{\sqrt{ax^2+bx+c}} + \frac{b}{2}\int \frac{dx}{x\sqrt{ax^2+bx+c}}$$

**14.290** $$\int \frac{dx}{(ax^2+bx+c)^{3/2}} = \frac{2(2ax+b)}{(4ac-b^2)\sqrt{ax^2+bx+c}}$$

**14.291** $$\int \frac{x\,dx}{(ax^2+bx+c)^{3/2}} = \frac{2(bx+2c)}{(b^2-4ac)\sqrt{ax^2+bx+c}}$$

**14.292** $$\int \frac{x^2\,dx}{(ax^2+bx+c)^{3/2}} = \frac{(2b^2-4ac)x+2bc}{a(4ac-b^2)\sqrt{ax^2+bx+c}} + \frac{1}{a}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

**14.293** $$\int \frac{dx}{x(ax^2+bx+c)^{3/2}} = \frac{1}{c\sqrt{ax^2+bx+c}} + \frac{1}{c}\int \frac{dx}{x\sqrt{ax^2+bx+c}} - \frac{b}{2c}\int \frac{dx}{(ax^2+bx+c)^{3/2}}$$

**14.294** $$\int \frac{dx}{x^2(ax^2+bx+c)^{3/2}} = -\frac{ax^2+2bx+c}{c^2x\sqrt{ax^2+bx+c}} + \frac{b^2-2ac}{2c^2}\int \frac{dx}{(ax^2+bx+c)^{3/2}} - \frac{3b}{2c^2}\int \frac{dx}{x\sqrt{ax^2+bx+c}}$$

**14.295** $$\int (ax^2+bx+c)^{n+1/2}\,dx = \frac{(2ax+b)(ax^2+bx+c)^{n+1/2}}{4a(n+1)} + \frac{(2n+1)(4ac-b^2)}{8a(n+1)}\int (ax^2+bx+c)^{n-1/2}\,dx$$

**14.296**
$$\int x(ax^2+bx+c)^{n+1/2}\,dx = \frac{(ax^2+bx+c)^{n+3/2}}{a(2n+3)} - \frac{b}{2a}\int (ax^2+bx+c)^{n+1/2}\,dx$$

**14.297**
$$\int \frac{dx}{(ax^2+bx+c)^{n+1/2}} = \frac{2(2ax+b)}{(2n-1)(4ac-b^2)(ax^2+bx+c)^{n-1/2}} + \frac{8a(n-1)}{(2n-1)(4ac-b^2)}\int \frac{dx}{(ax^2+bx+c)^{n-1/2}}$$

**14.298**
$$\int \frac{dx}{x(ax^2+bx+c)^{n+1/2}} = \frac{1}{(2n-1)c(ax^2+bx+c)^{n-1/2}} + \frac{1}{c}\int \frac{dx}{x(ax^2+bx+c)^{n-1/2}} - \frac{b}{2c}\int \frac{dx}{(ax^2+bx+c)^{n+1/2}}$$

## INTEGRALS INVOLVING $x^3 + a^3$

Note that for formulas involving $x^3 - a^3$ replace $a$ by $-a$.

**14.299**
$$\int \frac{dx}{x^3+a^3} = \frac{1}{6a^2}\ln\frac{(x+a)^2}{x^2-ax+a^2} + \frac{1}{a^2\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

**14.300**
$$\int \frac{x\,dx}{x^3+a^3} = \frac{1}{6a}\ln\frac{x^2-ax+a^2}{(x+a)^2} + \frac{1}{a\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

**14.301**
$$\int \frac{x^2\,dx}{x^3+a^3} = \frac{1}{3}\ln(x^3+a^3)$$

**14.302**
$$\int \frac{dx}{x(x^3+a^3)} = \frac{1}{3a^3}\ln\left(\frac{x^3}{x^3+a^3}\right)$$

**14.303**
$$\int \frac{dx}{x^2(x^3+a^3)} = -\frac{1}{a^3x} - \frac{1}{6a^4}\ln\frac{x^2-ax+a^2}{(x+a)^2} - \frac{1}{a^4\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

**14.304**
$$\int \frac{dx}{(x^3+a^3)^2} = \frac{x}{3a^3(x^3+a^3)} + \frac{1}{9a^5}\ln\frac{(x+a)^2}{x^2-ax+a^2} + \frac{2}{3a^5\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

**14.305**
$$\int \frac{x\,dx}{(x^3+a^3)^2} = \frac{x^2}{3a^3(x^3+a^3)} + \frac{1}{18a^4}\ln\frac{x^2-ax+a^2}{(x+a)^2} + \frac{1}{3a^4\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

**14.306**
$$\int \frac{x^2\,dx}{(x^3+a^3)^2} = -\frac{1}{3(x^3+a^3)}$$

**14.307**
$$\int \frac{dx}{x(x^3+a^3)^2} = \frac{1}{3a^3(x^3+a^3)} + \frac{1}{3a^6}\ln\left(\frac{x^3}{x^3+a^3}\right)$$

**14.308**
$$\int \frac{dx}{x^2(x^3+a^3)^2} = -\frac{1}{a^6x} - \frac{x^2}{3a^6(x^3+a^3)} - \frac{4}{3a^6}\int \frac{x\,dx}{x^3+a^3}$$
[See 14.300]

**14.309**
$$\int \frac{x^m\,dx}{x^3+a^3} = \frac{x^{m-2}}{m-2} - a^3\int \frac{x^{m-3}\,dx}{x^3+a^3}$$

**14.310**
$$\int \frac{dx}{x^n(x^3+a^3)} = \frac{-1}{a^3(n-1)x^{n-1}} - \frac{1}{a^3}\int \frac{dx}{x^{n-3}(x^3+a^3)}$$

## INTEGRALS INVOLVING $x^4 \pm a^4$

**14.311**
$$\int \frac{dx}{x^4+a^4} = \frac{1}{4a^3\sqrt{2}}\ln\left(\frac{x^2+ax\sqrt{2}+a^2}{x^2-ax\sqrt{2}+a^2}\right) - \frac{1}{2a^3\sqrt{2}}\tan^{-1}\frac{ax\sqrt{2}}{x^2-a^2}$$

**14.312**
$$\int \frac{x\,dx}{x^4+a^4} = \frac{1}{2a^2}\tan^{-1}\frac{x^2}{a^2}$$

**14.313**
$$\int \frac{x^2\,dx}{x^4+a^4} = \frac{1}{4a\sqrt{2}}\ln\left(\frac{x^2-ax\sqrt{2}+a^2}{x^2+ax\sqrt{2}+a^2}\right) - \frac{1}{2a\sqrt{2}}\tan^{-1}\frac{ax\sqrt{2}}{x^2-a^2}$$

**14.314**
$$\int \frac{x^3\,dx}{x^4+a^4} = \frac{1}{4}\ln(x^4+a^4)$$

**14.315**
$$\int \frac{dx}{x(x^4+a^4)} = \frac{1}{4a^4} \ln\left(\frac{x^4}{x^4+a^4}\right)$$

**14.316**
$$\int \frac{dx}{x^2(x^4+a^4)} = -\frac{1}{a^4x} - \frac{1}{4a^5\sqrt{2}} \ln\left(\frac{x^2-ax\sqrt{2}+a^2}{x^2+ax\sqrt{2}+a^2}\right) + \frac{1}{2a^5\sqrt{2}} \tan^{-1}\frac{ax\sqrt{2}}{x^2-a^2}$$

**14.317**
$$\int \frac{dx}{x^3(x^4+a^4)} = -\frac{1}{2a^4x^2} - \frac{1}{2a^6} \tan^{-1}\frac{x^2}{a^2}$$

**14.318**
$$\int \frac{dx}{x^4-a^4} = \frac{1}{4a^3} \ln\left(\frac{x-a}{x+a}\right) - \frac{1}{2a^3} \tan^{-1}\frac{x}{a}$$

**14.319**
$$\int \frac{x\,dx}{x^4-a^4} = \frac{1}{4a^2} \ln\left(\frac{x^2-a^2}{x^2+a^2}\right)$$

**14.320**
$$\int \frac{x^2\,dx}{x^4-a^4} = \frac{1}{4a} \ln\left(\frac{x-a}{x+a}\right) + \frac{1}{2a} \tan^{-1}\frac{x}{a}$$

**14.321**
$$\int \frac{x^3\,dx}{x^4-a^4} = \frac{1}{4} \ln(x^4-a^4)$$

**14.322**
$$\int \frac{dx}{x(x^4-a^4)} = \frac{1}{4a^4} \ln\left(\frac{x^4-a^4}{x^4}\right)$$

**14.323**
$$\int \frac{dx}{x^2(x^4-a^4)} = \frac{1}{a^4x} + \frac{1}{4a^5} \ln\left(\frac{x-a}{x+a}\right) + \frac{1}{2a^5} \tan^{-1}\frac{x}{a}$$

**14.324**
$$\int \frac{dx}{x^3(x^4-a^4)} = \frac{1}{2a^4x^2} + \frac{1}{4a^6} \ln\left(\frac{x^2-a^2}{x^2+a^2}\right)$$

## INTEGRALS INVOLVING $x^n \pm a^n$

**14.325**
$$\int \frac{dx}{x(x^n+a^n)} = \frac{1}{na^n} \ln \frac{x^n}{x^n+a^n}$$

**14.326**
$$\int \frac{x^{n-1}\,dx}{x^n+a^n} = \frac{1}{n} \ln(x^n+a^n)$$

**14.327**
$$\int \frac{x^m\,dx}{(x^n+a^n)^r} = \int \frac{x^{m-n}\,dx}{(x^n+a^n)^{r-1}} - a^n \int \frac{x^{m-n}\,dx}{(x^n+a^n)^r}$$

**14.328**
$$\int \frac{dx}{x^m(x^n+a^n)^r} = \frac{1}{a^n} \int \frac{dx}{x^m(x^n+a^n)^{r-1}} - \frac{1}{a^n} \int \frac{dx}{x^{m-n}(x^n+a^n)^r}$$

**14.329**
$$\int \frac{dx}{x\sqrt{x^n+a^n}} = \frac{1}{n\sqrt{a^n}} \ln\left(\frac{\sqrt{x^n+a^n}-\sqrt{a^n}}{\sqrt{x^n+a^n}+\sqrt{a^n}}\right)$$

**14.330**
$$\int \frac{dx}{x(x^n-a^n)} = \frac{1}{na^n} \ln\left(\frac{x^n-a^n}{x^n}\right)$$

**14.331**
$$\int \frac{x^{n-1}\,dx}{x^n-a^n} = \frac{1}{n} \ln(x^n-a^n)$$

**14.332**
$$\int \frac{x^m\,dx}{(x^n-a^n)^r} = a^n \int \frac{x^{m-n}\,dx}{(x^n-a^n)^r} + \int \frac{x^{m-n}\,dx}{(x^n-a^n)^{r-1}}$$

**14.333**
$$\int \frac{dx}{x^m(x^n-a^n)^r} = \frac{1}{a^n} \int \frac{dx}{x^{m-n}(x^n-a^n)^r} - \frac{1}{a^n} \int \frac{dx}{x^m(x^n-a^n)^{r-1}}$$

**14.334**
$$\int \frac{dx}{x\sqrt{x^n-a^n}} = \frac{2}{n\sqrt{a^n}} \cos^{-1}\sqrt{\frac{a^n}{x^n}}$$

**14.335**
$$\begin{aligned}\int \frac{x^{p-1}\,dx}{x^{2m}+a^{2m}} &= \frac{1}{ma^{2m-p}}\sum_{k=1}^{m}\sin\frac{(2k-1)p\pi}{2m}\tan^{-1}\left(\frac{x+a\cos[(2k-1)\pi/2m]}{a\sin[(2k-1)\pi/2m]}\right)\\ &\quad -\frac{1}{2ma^{2m-p}}\sum_{k=1}^{m}\cos\frac{(2k-1)p\pi}{2m}\ln\left(x^2+2ax\cos\frac{(2k-1)\pi}{2m}+a^2\right)\end{aligned}$$
where $0<p\leqq 2m$.

**14.336**
$$\begin{aligned}\int \frac{x^{p-1}\,dx}{x^{2m}-a^{2m}} &= \frac{1}{2ma^{2m-p}}\sum_{k=1}^{m-1}\cos\frac{kp\pi}{m}\ln\left(x^2-2ax\cos\frac{k\pi}{m}+a^2\right)\\ &\quad -\frac{1}{ma^{2m-p}}\sum_{k=1}^{m-1}\sin\frac{kp\pi}{m}\tan^{-1}\left(\frac{x-a\cos(k\pi/m)}{a\sin(k\pi/m)}\right)\\ &\quad +\frac{1}{2ma^{2m-p}}\{\ln(x-a)+(-1)^p\ln(x+a)\}\end{aligned}$$
where $0<p\leqq 2m$.

**14.337**
$$\begin{aligned}&\int \frac{x^{p-1}\,dx}{x^{2m+1}+a^{2m+1}}\\ &= \frac{2(-1)^{p-1}}{(2m+1)a^{2m-p+1}}\sum_{k=1}^{m}\sin\frac{2kp\pi}{2m+1}\tan^{-1}\left(\frac{x+a\cos[2k\pi/(2m+1)]}{a\sin[2k\pi/(2m+1)]}\right)\\ &\quad -\frac{(-1)^{p-1}}{(2m+1)a^{2m-p+1}}\sum_{k=1}^{m}\cos\frac{2kp\pi}{2m+1}\ln\left(x^2+2ax\cos\frac{2k\pi}{2m+1}+a^2\right)\\ &\quad +\frac{(-1)^{p-1}\ln(x+a)}{(2m+1)a^{2m-p+1}}\end{aligned}$$
where $0<p\leqq 2m+1$.

**14.338**
$$\begin{aligned}&\int \frac{x^{p-1}\,dx}{x^{2m+1}-a^{2m+1}}\\ &= \frac{-2}{(2m+1)a^{2m-p+1}}\sum_{k=1}^{m}\sin\frac{2kp\pi}{2m+1}\tan^{-1}\left(\frac{x-a\cos[2k\pi/(2m+1)]}{a\sin[2k\pi/(2m+1)]}\right)\\ &\quad +\frac{1}{(2m+1)a^{2m-p+1}}\sum_{k=1}^{m}\cos\frac{2kp\pi}{2m+1}\ln\left(x^2-2ax\cos\frac{2k\pi}{2m+1}+a^2\right)\\ &\quad +\frac{\ln(x-a)}{(2m+1)a^{2m-p+1}}\end{aligned}$$
where $0<p\leqq 2m+1$.

## INTEGRALS INVOLVING sin $ax$

**14.339** $$\int \sin ax\,dx = -\frac{\cos ax}{a}$$

**14.340** $$\int x\sin ax\,dx = \frac{\sin ax}{a^2}-\frac{x\cos ax}{a}$$

**14.341** $$\int x^2\sin ax\,dx = \frac{2x}{a^2}\sin ax+\left(\frac{2}{a^3}-\frac{x^2}{a}\right)\cos ax$$

**14.342** $$\int x^3\sin ax\,dx = \left(\frac{3x^2}{a^2}-\frac{6}{a^4}\right)\sin ax+\left(\frac{6x}{a^3}-\frac{x^3}{a}\right)\cos ax$$

**14.343** $$\int \frac{\sin ax}{x}\,dx = ax-\frac{(ax)^3}{3\cdot 3!}+\frac{(ax)^5}{5\cdot 5!}-\cdots$$

**14.344** $$\int \frac{\sin ax}{x^2}\,dx = -\frac{\sin ax}{x}+a\int\frac{\cos ax}{x}\,dx \qquad \text{[see 14.373]}$$

**14.345** $$\int \frac{dx}{\sin ax} = \frac{1}{a}\ln(\csc ax-\cot ax) = \frac{1}{a}\ln\tan\frac{ax}{2}$$

**14.346** $$\int \frac{x\,dx}{\sin ax} = \frac{1}{a^2}\left\{ax+\frac{(ax)^3}{18}+\frac{7(ax)^5}{1800}+\cdots+\frac{2(2^{2n-1}-1)B_n(ax)^{2n+1}}{(2n+1)!}+\cdots\right\}$$

**14.347** $$\int \sin^2 ax\,dx = \frac{x}{2}-\frac{\sin 2ax}{4a}$$

**14.348** $$\int x\sin^2 ax\,dx = \frac{x^2}{4} - \frac{x\sin 2ax}{4a} - \frac{\cos 2ax}{8a^2}$$

**14.349** $$\int \sin^3 ax\,dx = -\frac{\cos ax}{a} + \frac{\cos^3 ax}{3a}$$

**14.350** $$\int \sin^4 ax\,dx = \frac{3x}{8} - \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$$

**14.351** $$\int \frac{dx}{\sin^2 ax} = -\frac{1}{a}\cot ax$$

**14.352** $$\int \frac{dx}{\sin^3 ax} = -\frac{\cos ax}{2a\sin^2 ax} + \frac{1}{2a}\ln\tan\frac{ax}{2}$$

**14.353** $$\int \sin px\sin qx\,dx = \frac{\sin(p-q)x}{2(p-q)} - \frac{\sin(p+q)x}{2(p+q)}$$ [If $p = \pm q$, see 14.368.]

**14.354** $$\int \frac{dx}{1-\sin ax} = \frac{1}{a}\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)$$

**14.355** $$\int \frac{x\,dx}{1-\sin ax} = \frac{x}{a}\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right) + \frac{2}{a^2}\ln\sin\left(\frac{\pi}{4}-\frac{ax}{2}\right)$$

**14.356** $$\int \frac{dx}{1+\sin ax} = -\frac{1}{a}\tan\left(\frac{\pi}{4}-\frac{ax}{2}\right)$$

**14.357** $$\int \frac{x\,dx}{1+\sin ax} = -\frac{x}{a}\tan\left(\frac{\pi}{4}-\frac{ax}{2}\right) + \frac{2}{a^2}\ln\sin\left(\frac{\pi}{4}+\frac{ax}{2}\right)$$

**14.358** $$\int \frac{dx}{(1-\sin ax)^2} = \frac{1}{2a}\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right) + \frac{1}{6a}\tan^3\left(\frac{\pi}{4}+\frac{ax}{2}\right)$$

**14.359** $$\int \frac{dx}{(1+\sin ax)^2} = -\frac{1}{2a}\tan\left(\frac{\pi}{4}-\frac{ax}{2}\right) - \frac{1}{6a}\tan^3\left(\frac{\pi}{4}-\frac{ax}{2}\right)$$

**14.360** $$\int \frac{dx}{p+q\sin ax} = \begin{cases} \dfrac{2}{a\sqrt{p^2-q^2}}\tan^{-1}\dfrac{p\tan\frac{1}{2}ax+q}{\sqrt{p^2-q^2}} \\[2ex] \dfrac{1}{a\sqrt{q^2-p^2}}\ln\left(\dfrac{p\tan\frac{1}{2}ax+q-\sqrt{q^2-p^2}}{p\tan\frac{1}{2}ax+q+\sqrt{q^2-p^2}}\right) \end{cases}$$

If $p = \pm q$ see 14.354 and 14.356.

**14.361** $$\int \frac{dx}{(p+q\sin ax)^2} = \frac{q\cos ax}{a(p^2-q^2)(p+q\sin ax)} + \frac{p}{p^2-q^2}\int\frac{dx}{p+q\sin ax}$$

If $p = \pm q$ see 14.358 and 14.359.

**14.362** $$\int \frac{dx}{p^2+q^2\sin^2 ax} = \frac{1}{ap\sqrt{p^2+q^2}}\tan^{-1}\frac{\sqrt{p^2+q^2}\tan ax}{p}$$

**14.363** $$\int \frac{dx}{p^2-q^2\sin^2 ax} = \begin{cases} \dfrac{1}{ap\sqrt{p^2-q^2}}\tan^{-1}\dfrac{\sqrt{p^2-q^2}\tan ax}{p} \\[2ex] \dfrac{1}{2ap\sqrt{q^2-p^2}}\ln\left(\dfrac{\sqrt{q^2-p^2}\tan ax+p}{\sqrt{q^2-p^2}\tan ax-p}\right) \end{cases}$$

**14.364** $$\int x^m\sin ax\,dx = -\frac{x^m\cos ax}{a} + \frac{mx^{m-1}\sin ax}{a^2} - \frac{m(m-1)}{a^2}\int x^{m-2}\sin ax\,dx$$

**14.365** $$\int \frac{\sin ax}{x^n}\,dx = -\frac{\sin ax}{(n-1)x^{n-1}} + \frac{a}{n-1}\int\frac{\cos ax}{x^{n-1}}\,dx$$ [see 14.395]

**14.366** $$\int \sin^n ax\,dx = -\frac{\sin^{n-1}ax\cos ax}{an} + \frac{n-1}{n}\int\sin^{n-2}ax\,dx$$

**14.367** $$\int \frac{dx}{\sin^n ax} = \frac{-\cos ax}{a(n-1)\sin^{n-1}ax} + \frac{n-2}{n-1}\int\frac{dx}{\sin^{n-2}ax}$$

**14.368** $$\int \frac{x\,dx}{\sin^n ax} = \frac{-x\cos ax}{a(n-1)\sin^{n-1}ax} - \frac{1}{a^2(n-1)(n-2)\sin^{n-2}ax} + \frac{n-2}{n-1}\int\frac{x\,dx}{\sin^{n-2}ax}$$

## INTEGRALS INVOLVING $\cos ax$

**14.369** $\int \cos ax\,dx = \frac{\sin ax}{a}$

**14.370** $\int x\cos ax\,dx = \frac{\cos ax}{a^2} + \frac{x\sin ax}{a}$

**14.371** $\int x^2\cos ax\,dx = \frac{2x}{a^2}\cos ax + \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\sin ax$

**14.372** $\int x^3\cos ax\,dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4}\right)\cos ax + \left(\frac{x^3}{a} - \frac{6x}{a^3}\right)\sin ax$

**14.373** $\int \frac{\cos ax}{x}\,dx = \ln x - \frac{(ax)^2}{2\cdot 2!} + \frac{(ax)^4}{4\cdot 4!} - \frac{(ax)^6}{6\cdot 6!} + \cdots$

**14.374** $\int \frac{\cos ax}{x^2}\,dx = -\frac{\cos ax}{x} - a\int \frac{\sin ax}{x}\,dx$ [See 14.343]

**14.375** $\int \frac{dx}{\cos ax} = \frac{1}{a}\ln(\sec ax + \tan ax) = \frac{1}{a}\ln\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$

**14.376** $\int \frac{x\,dx}{\cos ax} = \frac{1}{a^2}\left\{\frac{(ax)^2}{2} + \frac{(ax)^4}{8} + \frac{5(ax)^6}{144} + \cdots + \frac{E_n(ax)^{2n+2}}{(2n+2)(2n)!} + \cdots\right\}$

**14.377** $\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$

**14.378** $\int x\cos^2 ax\,dx = \frac{x^2}{4} + \frac{x\sin 2ax}{4a} + \frac{\cos 2ax}{8a^2}$

**14.379** $\int \cos^3 ax\,dx = \frac{\sin ax}{a} - \frac{\sin^3 ax}{3a}$

**14.380** $\int \cos^4 ax\,dx = \frac{3x}{8} + \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$

**14.381** $\int \frac{dx}{\cos^2 ax} = \frac{\tan ax}{a}$

**14.382** $\int \frac{dx}{\cos^3 ax} = \frac{\sin ax}{2a\cos^2 ax} + \frac{1}{2a}\ln\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$

**14.383** $\int \cos ax\cos px\,dx = \frac{\sin(a-p)x}{2(a-p)} + \frac{\sin(a+p)x}{2(a+p)}$ [If $a = \pm p$, see 14.377.]

**14.384** $\int \frac{dx}{1 - \cos ax} = -\frac{1}{a}\cot\frac{ax}{2}$

**14.385** $\int \frac{x\,dx}{1 - \cos ax} = -\frac{x}{a}\cot\frac{ax}{2} + \frac{2}{a^2}\ln\sin\frac{ax}{2}$

**14.386** $\int \frac{dx}{1 + \cos ax} = \frac{1}{a}\tan\frac{ax}{2}$

**14.387** $\int \frac{x\,dx}{1 + \cos ax} = \frac{x}{a}\tan\frac{ax}{2} + \frac{2}{a^2}\ln\cos\frac{ax}{2}$

**14.388** $\int \frac{dx}{(1 - \cos ax)^2} = -\frac{1}{2a}\cot\frac{ax}{2} - \frac{1}{6a}\cot^3\frac{ax}{2}$

**14.389** $\int \frac{dx}{(1 + \cos ax)^2} = \frac{1}{2a}\tan\frac{ax}{2} + \frac{1}{6a}\tan^3\frac{ax}{2}$

**14.390** $$\int \frac{dx}{p+q\cos ax} = \begin{cases} \dfrac{2}{a\sqrt{p^2-q^2}}\tan^{-1}\sqrt{(p-q)/(p+q)}\,\tan\tfrac{1}{2}ax \\[2ex] \dfrac{1}{a\sqrt{q^2-p^2}}\ln\left(\dfrac{\tan\frac{1}{2}ax+\sqrt{(q+p)/(q-p)}}{\tan\frac{1}{2}ax-\sqrt{(q+p)/(q-p)}}\right) \end{cases}$$

[If $p = \pm q$ see 14.384 and 14.386.]

**14.391** $$\int \frac{dx}{(p+q\cos ax)^2} = \frac{q\sin ax}{a(q^2-p^2)(p+q\cos ax)} - \frac{p}{q^2-p^2}\int\frac{dx}{p+q\cos ax}$$

[If $p = \pm q$ see 14.388 and 14.389.]

**14.392** $$\int \frac{dx}{p^2+q^2\cos^2 ax} = \frac{1}{ap\sqrt{p^2+q^2}}\tan^{-1}\frac{p\tan ax}{\sqrt{p^2+q^2}}$$

**14.393** $$\int \frac{dx}{p^2-q^2\cos^2 ax} = \begin{cases} \dfrac{1}{ap\sqrt{p^2-q^2}}\tan^{-1}\dfrac{p\tan ax}{\sqrt{p^2-q^2}} \\[2ex] \dfrac{1}{2ap\sqrt{q^2-p^2}}\ln\left(\dfrac{p\tan ax-\sqrt{q^2-p^2}}{p\tan ax+\sqrt{q^2-p^2}}\right) \end{cases}$$

**14.394** $$\int x^m\cos ax\,dx = \frac{x^m\sin ax}{a} + \frac{mx^{m-1}}{a^2}\cos ax - \frac{m(m-1)}{a^2}\int x^{m-2}\cos ax\,dx$$

**14.395** $$\int\frac{\cos ax}{x^n}\,dx = -\frac{\cos ax}{(n-1)x^{n-1}} - \frac{a}{n-1}\int\frac{\sin ax}{x^{n-1}}\,dx$$

[See 14.365]

**14.396** $$\int \cos^n ax\,dx = \frac{\sin ax\cos^{n-1}ax}{an} + \frac{n-1}{n}\int\cos^{n-2}ax\,dx$$

**14.397** $$\int\frac{dx}{\cos^n ax} = \frac{\sin ax}{a(n-1)\cos^{n-1}ax} + \frac{n-2}{n-1}\int\frac{dx}{\cos^{n-2}ax}$$

**14.398** $$\int\frac{x\,dx}{\cos^n ax} = \frac{x\sin ax}{a(n-1)\cos^{n-1}ax} - \frac{1}{a^2(n-1)(n-2)\cos^{n-2}ax} + \frac{n-2}{n-1}\int\frac{x\,dx}{\cos^{n-2}ax}$$

## INTEGRALS INVOLVING sin *ax* AND cos *ax*

**14.399** $$\int\sin ax\cos ax\,dx = \frac{\sin^2 ax}{2a}$$

**14.400** $$\int\sin px\cos qx\,dx = -\frac{\cos(p-q)x}{2(p-q)} - \frac{\cos(p+q)x}{2(p+q)}$$

**14.401** $$\int\sin^n ax\cos ax\,dx = \frac{\sin^{n+1}ax}{(n+1)a}$$

[If $n = -1$, see 14.440.]

**14.402** $$\int\cos^n ax\sin ax\,dx = -\frac{\cos^{n+1}ax}{(n+1)a}$$

[If $n = -1$, see 14.429.]

**14.403** $$\int\sin^2 ax\cos^2 ax\,dx = \frac{x}{8} - \frac{\sin 4ax}{32a}$$

**14.404** $$\int\frac{dx}{\sin ax\cos ax} = \frac{1}{a}\ln\tan ax$$

**14.405** $$\int\frac{dx}{\sin^2 ax\cos ax} = \frac{1}{a}\ln\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right) - \frac{1}{a\sin ax}$$

**14.406** $$\int\frac{dx}{\sin ax\cos^2 ax} = \frac{1}{a}\ln\tan\frac{ax}{2} + \frac{1}{a\cos ax}$$

**14.407** $$\int\frac{dx}{\sin^2 ax\cos^2 ax} = -\frac{2\cot 2ax}{a}$$

**14.408**
$$\int \frac{\sin^2 ax}{\cos ax}\,dx = -\frac{\sin ax}{a} + \frac{1}{a}\ln\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)$$

**14.409**
$$\int \frac{\cos^2 ax}{\sin ax}\,dx = \frac{\cos ax}{a} + \frac{1}{a}\ln\tan\frac{ax}{2}$$

**14.410**
$$\int \frac{dx}{\cos ax(1 \pm \sin ax)} = \mp\frac{1}{2a(1 \pm \sin ax)} + \frac{1}{2a}\ln\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)$$

**14.411**
$$\int \frac{dx}{\sin ax(1 \pm \cos ax)} = \pm\frac{1}{2a(1 \pm \cos ax)} + \frac{1}{2a}\ln\tan\frac{ax}{2}$$

**14.412**
$$\int \frac{dx}{\sin ax \pm \cos ax} = \frac{1}{a\sqrt{2}}\ln\tan\left(\frac{ax}{2}\pm\frac{\pi}{8}\right)$$

**14.413**
$$\int \frac{\sin ax\,dx}{\sin ax \pm \cos ax} = \frac{x}{2} \mp \frac{1}{2a}\ln(\sin ax \pm \cos ax)$$

**14.414**
$$\int \frac{\cos ax\,dx}{\sin ax \pm \cos ax} = \pm\frac{x}{2} + \frac{1}{2a}\ln(\sin ax \pm \cos ax)$$

**14.415**
$$\int \frac{\sin ax\,dx}{p + q\cos ax} = -\frac{1}{aq}\ln(p + q\cos ax)$$

**14.416**
$$\int \frac{\cos ax\,dx}{p + q\sin ax} = \frac{1}{aq}\ln(p + q\sin ax)$$

**14.417**
$$\int \frac{\sin ax\,dx}{(p + q\cos ax)^n} = \frac{1}{aq(n-1)(p+q\cos ax)^{n-1}}$$

**14.418**
$$\int \frac{\cos ax\,dx}{(p + q\sin ax)^n} = \frac{-1}{aq(n-1)(p+q\sin ax)^{n-1}}$$

**14.419**
$$\int \frac{dx}{p\sin ax + q\cos ax} = \frac{1}{a\sqrt{p^2+q^2}}\ln\tan\left(\frac{ax+\tan^{-1}(q/p)}{2}\right)$$

**14.420**
$$\int \frac{dx}{p\sin ax + q\cos ax + r} = \begin{cases} \dfrac{2}{a\sqrt{r^2-p^2-q^2}}\tan^{-1}\left(\dfrac{p+(r-q)\tan(ax/2)}{\sqrt{r^2-p^2-q^2}}\right) \\[2ex] \dfrac{1}{a\sqrt{p^2+q^2-r^2}}\ln\left(\dfrac{p-\sqrt{p^2+q^2-r^2}+(r-q)\tan(ax/2)}{p+\sqrt{p^2+q^2-r^2}+(r-q)\tan(ax/2)}\right) \end{cases}$$

If $r = q$ see 14.421. If $r^2 = p^2 + q^2$ see 14.422.

**14.421**
$$\int \frac{dx}{p\sin ax + q(1+\cos ax)} = \frac{1}{ap}\ln\left(q + p\tan\frac{ax}{2}\right)$$

**14.422**
$$\int \frac{dx}{p\sin ax + q\cos ax \pm \sqrt{p^2+q^2}} = \frac{-1}{a\sqrt{p^2+q^2}}\tan\left(\frac{\pi}{4}\mp\frac{ax+\tan^{-1}(q/p)}{2}\right)$$

**14.423**
$$\int \frac{dx}{p^2\sin^2 ax + q^2\cos^2 ax} = \frac{1}{apq}\tan^{-1}\left(\frac{p\tan ax}{q}\right)$$

**14.424**
$$\int \frac{dx}{p^2\sin^2 ax - q^2\cos^2 ax} = \frac{1}{2apq}\ln\left(\frac{p\tan ax - q}{p\tan ax + q}\right)$$

**14.425**
$$\int \sin^m ax\cos^n ax\,dx = \begin{cases} -\dfrac{\sin^{m-1}ax\cos^{n+1}ax}{a(m+n)} + \dfrac{m-1}{m+n}\displaystyle\int \sin^{m-2}ax\cos^n ax\,dx \\[2ex] \dfrac{\sin^{m+1}ax\cos^{n-1}ax}{a(m+n)} + \dfrac{n-1}{m+n}\displaystyle\int \sin^m ax\cos^{n-2}ax\,dx \end{cases}$$

**14.426** $$\int \frac{\sin^m ax}{\cos^n ax}\,dx = \begin{cases} \dfrac{\sin^{m-1} ax}{a(n-1)\cos^{n-1} ax} - \dfrac{m-1}{n-1}\displaystyle\int \frac{\sin^{m-2} ax}{\cos^{n-2} ax}\,dx \\ \dfrac{\sin^{m+1} ax}{a(n-1)\cos^{n-1} ax} - \dfrac{m-n+2}{n-1}\displaystyle\int \frac{\sin^{m} ax}{\cos^{n-2} ax}\,dx \\ \dfrac{-\sin^{m-1} ax}{a(m-n)\cos^{n-1} ax} + \dfrac{m-1}{m-n}\displaystyle\int \frac{\sin^{m-2} ax}{\cos^{n} ax}\,dx \end{cases}$$

**14.427** $$\int \frac{\cos^m ax}{\sin^n ax}\,dx = \begin{cases} \dfrac{-\cos^{m-1} ax}{a(n-1)\sin^{n-1} ax} - \dfrac{m-1}{n-1}\displaystyle\int \frac{\cos^{m-2} ax}{\sin^{n-2} ax}\,dx \\ \dfrac{-\cos^{m+1} ax}{a(n-1)\sin^{n-1} ax} - \dfrac{m-n+2}{n-1}\displaystyle\int \frac{\cos^{m} ax}{\sin^{n-2} ax}\,dx \\ \dfrac{\cos^{m-1} ax}{a(m-n)\sin^{n-1} ax} + \dfrac{m-1}{m-n}\displaystyle\int \frac{\cos^{m-2} ax}{\sin^{n} ax}\,dx \end{cases}$$

**14.428** $$\int \frac{dx}{\sin^m ax \cos^n ax} = \begin{cases} \dfrac{1}{a(n-1)\sin^{m-1} ax \cos^{n-1} ax} + \dfrac{m+n-2}{n-1}\displaystyle\int \frac{dx}{\sin^m ax \cos^{n-2} ax} \\ \dfrac{-1}{a(m-1)\sin^{m-1} ax \cos^{n-1} ax} + \dfrac{m+n-2}{m-1}\displaystyle\int \frac{dx}{\sin^{m-2} ax \cos^{n} ax} \end{cases}$$

## INTEGRALS INVOLVING $\tan ax$

**14.429** $$\int \tan ax\,dx = -\frac{1}{a}\ln\cos ax = \frac{1}{a}\ln\sec ax$$

**14.430** $$\int \tan^2 ax\,dx = \frac{\tan ax}{a} - x$$

**14.431** $$\int \tan^3 ax\,dx = \frac{\tan^2 ax}{2a} + \frac{1}{a}\ln\cos ax$$

**14.432** $$\int \tan^n ax \sec^2 ax\,dx = \frac{\tan^{n+1} ax}{(n+1)a}$$

**14.433** $$\int \frac{\sec^2 ax}{\tan ax}\,dx = \frac{1}{a}\ln\tan ax$$

**14.434** $$\int \frac{dx}{\tan ax} = \frac{1}{a}\ln\sin ax$$

**14.435** $$\int x\tan ax\,dx = \frac{1}{a^2}\left\{\frac{(ax)^3}{3} + \frac{(ax)^5}{15} + \frac{2(ax)^7}{105} + \cdots + \frac{2^{2n}(2^{2n}-1)B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$$

**14.436** $$\int \frac{\tan ax}{x}\,dx = ax + \frac{(ax)^3}{9} + \frac{2(ax)^5}{75} + \cdots + \frac{2^{2n}(2^{2n}-1)B_n(ax)^{2n-1}}{(2n-1)(2n)!} + \cdots$$

**14.437** $$\int x\tan^2 ax\,dx = \frac{x\tan ax}{a} + \frac{1}{a^2}\ln\cos ax - \frac{x^2}{2}$$

**14.438** $$\int \frac{dx}{p + q\tan ax} = \frac{px}{p^2+q^2} + \frac{q}{a(p^2+q^2)}\ln(q\sin ax + p\cos ax)$$

**14.439** $$\int \tan^n ax\,dx = \frac{\tan^{n-1} ax}{(n-1)a} - \int \tan^{n-2} ax\,dx$$

## INTEGRALS INVOLVING $\cot ax$

**14.440** $\displaystyle\int \cot ax\,dx = \frac{1}{a}\ln\sin ax$

**14.441** $\displaystyle\int \cot^2 ax\,dx = -\frac{\cot ax}{a} - x$

**14.442** $\displaystyle\int \cot^3 ax\,dx = -\frac{\cot^2 ax}{2a} - \frac{1}{a}\ln\sin ax$

**14.443** $\displaystyle\int \cot^n ax\csc^2 ax\,dx = -\frac{\cot^{n+1} ax}{(n+1)a}$

**14.444** $\displaystyle\int \frac{\csc^2 ax}{\cot ax}\,dx = -\frac{1}{a}\ln\cot ax$

**14.445** $\displaystyle\int \frac{dx}{\cot ax} = -\frac{1}{a}\ln\cos ax$

**14.446** $\displaystyle\int x\cot ax\,dx = \frac{1}{a^2}\left\{ax - \frac{(ax)^3}{9} - \frac{(ax)^5}{225} - \cdots - \frac{2^{2n}B_n(ax)^{2n+1}}{(2n+1)!} - \cdots\right\}$

**14.447** $\displaystyle\int \frac{\cot ax}{x}\,dx = -\frac{1}{ax} - \frac{ax}{3} - \frac{(ax)^3}{135} - \cdots - \frac{2^{2n}B_n(ax)^{2n-1}}{(2n-1)(2n)!} - \cdots$

**14.448** $\displaystyle\int x\cot^2 ax\,dx = -\frac{x\cot ax}{a} + \frac{1}{a^2}\ln\sin ax - \frac{x^2}{2}$

**14.449** $\displaystyle\int \frac{dx}{p + q\cot ax} = \frac{px}{p^2+q^2} - \frac{q}{a(p^2+q^2)}\ln(p\sin ax + q\cos ax)$

**14.450** $\displaystyle\int \cot^n ax\,dx = -\frac{\cot^{n-1} ax}{(n-1)a} - \int \cot^{n-2} ax\,dx$

## INTEGRALS INVOLVING $\sec ax$

**14.451** $\displaystyle\int \sec ax\,dx = \frac{1}{a}\ln(\sec ax + \tan ax) = \frac{1}{a}\ln\tan\left(\frac{ax}{2} + \frac{\pi}{4}\right)$

**14.452** $\displaystyle\int \sec^2 ax\,dx = \frac{\tan ax}{a}$

**14.453** $\displaystyle\int \sec^3 ax\,dx = \frac{\sec ax\tan ax}{2a} + \frac{1}{2a}\ln(\sec ax + \tan ax)$

**14.454** $\displaystyle\int \sec^n ax\tan ax\,dx = \frac{\sec^n ax}{na}$

**14.455** $\displaystyle\int \frac{dx}{\sec ax} = \frac{\sin ax}{a}$

**14.456** $\displaystyle\int x\sec ax\,dx = \frac{1}{a^2}\left\{\frac{(ax)^2}{2} + \frac{(ax)^4}{8} + \frac{5(ax)^6}{144} + \cdots + \frac{E_n(ax)^{2n+2}}{(2n+2)(2n)!} + \cdots\right\}$

**14.457** $\displaystyle\int \frac{\sec ax}{x}\,dx = \ln x + \frac{(ax)^2}{4} + \frac{5(ax)^4}{96} + \frac{61(ax)^6}{4320} + \cdots + \frac{E_n(ax)^{2n}}{2n(2n)!} + \cdots$

**14.458** $\displaystyle\int x\sec^2 ax\,dx = \frac{x}{a}\tan ax + \frac{1}{a^2}\ln\cos ax$

**14.459** $\displaystyle \int \frac{dx}{q + p \sec ax} = \frac{x}{q} - \frac{p}{q} \int \frac{dx}{p + q \cos ax}$

**14.460** $\displaystyle \int \sec^n ax\, dx = \frac{\sec^{n-2} ax \tan ax}{a(n-1)} + \frac{n-2}{n-1} \int \sec^{n-2} ax\, dx$

## INTEGRALS INVOLVING csc *ax*

**14.461** $\displaystyle \int \csc ax\, dx = \frac{1}{a} \ln (\csc ax - \cot ax) = \frac{1}{a} \ln \tan \frac{ax}{2}$

**14.462** $\displaystyle \int \csc^2 ax\, dx = -\frac{\cot ax}{a}$

**14.463** $\displaystyle \int \csc^3 ax\, dx = -\frac{\csc ax \cot ax}{2a} + \frac{1}{2a} \ln \tan \frac{ax}{2}$

**14.464** $\displaystyle \int \csc^n ax \cot ax\, dx = -\frac{\csc^n ax}{na}$

**14.465** $\displaystyle \int \frac{dx}{\csc ax} = -\frac{\cos ax}{a}$

**14.466** $\displaystyle \int x \csc ax\, dx = \frac{1}{a^2} \left\{ ax + \frac{(ax)^3}{18} + \frac{7(ax)^5}{1800} + \cdots + \frac{2(2^{2n-1} - 1) B_n (ax)^{2n+1}}{(2n+1)!} + \cdots \right\}$

**14.467** $\displaystyle \int \frac{\csc ax}{x}\, dx = -\frac{1}{ax} + \frac{ax}{6} + \frac{7(ax)^3}{1080} + \cdots + \frac{2(2^{2n-1} - 1) B_n (ax)^{2n-1}}{(2n-1)(2n)!} + \cdots$

**14.468** $\displaystyle \int x \csc^2 ax\, dx = -\frac{x \cot ax}{a} + \frac{1}{a^2} \ln \sin ax$

**14.469** $\displaystyle \int \frac{dx}{q + p \csc ax} = \frac{x}{q} - \frac{p}{q} \int \frac{dx}{p + q \sin ax}$ [See 14.360]

**14.470** $\displaystyle \int \csc^n ax\, dx = -\frac{\csc^{n-2} ax \cot ax}{a(n-1)} + \frac{n-2}{n-1} \int \csc^{n-2} ax\, dx$

## INTEGRALS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

**14.471** $\displaystyle \int \sin^{-1} \frac{x}{a}\, dx = x \sin^{-1} \frac{x}{a} + \sqrt{a^2 - x^2}$

**14.472** $\displaystyle \int x \sin^{-1} \frac{x}{a}\, dx = \left( \frac{x^2}{2} - \frac{a^2}{4} \right) \sin^{-1} \frac{x}{a} + \frac{x\sqrt{a^2 - x^2}}{4}$

**14.473** $\displaystyle \int x^2 \sin^{-1} \frac{x}{a}\, dx = \frac{x^3}{3} \sin^{-1} \frac{x}{a} + \frac{(x^2 + 2a^2)\sqrt{a^2 - x^2}}{9}$

**14.474** $\displaystyle \int \frac{\sin^{-1}(x/a)}{x}\, dx = \frac{x}{a} + \frac{(x/a)^3}{2 \cdot 3 \cdot 3} + \frac{1 \cdot 3 (x/a)^5}{2 \cdot 4 \cdot 5 \cdot 5} + \frac{1 \cdot 3 \cdot 5 (x/a)^7}{2 \cdot 4 \cdot 6 \cdot 7 \cdot 7} + \cdots$

**14.475** $\displaystyle \int \frac{\sin^{-1}(x/a)}{x^2}\, dx = -\frac{\sin^{-1}(x/a)}{x} - \frac{1}{a} \ln \left( \frac{a + \sqrt{a^2 - x^2}}{x} \right)$

**14.476** $\displaystyle \int \left( \sin^{-1} \frac{x}{a} \right)^2 dx = x \left( \sin^{-1} \frac{x}{a} \right)^2 - 2x + 2\sqrt{a^2 - x^2} \sin^{-1} \frac{x}{a}$

**14.477** $$\int \cos^{-1}\frac{x}{a}\,dx = x\cos^{-1}\frac{x}{a} - \sqrt{a^2-x^2}$$

**14.478** $$\int x\cos^{-1}\frac{x}{a}\,dx = \left(\frac{x^2}{2}-\frac{a^2}{4}\right)\cos^{-1}\frac{x}{a} - \frac{x\sqrt{a^2-x^2}}{4}$$

**14.479** $$\int x^2\cos^{-1}\frac{x}{a}\,dx = \frac{x^3}{3}\cos^{-1}\frac{x}{a} - \frac{(x^2+2a^2)\sqrt{a^2-x^2}}{9}$$

**14.480** $$\int \frac{\cos^{-1}(x/a)}{x}\,dx = \frac{\pi}{2}\ln x - \int \frac{\sin^{-1}(x/a)}{x}\,dx$$ [See 14.474]

**14.481** $$\int \frac{\cos^{-1}(x/a)}{x^2}\,dx = -\frac{\cos^{-1}(x/a)}{x} + \frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

**14.482** $$\int \left(\cos^{-1}\frac{x}{a}\right)^2 dx = x\left(\cos^{-1}\frac{x}{a}\right)^2 - 2x - 2\sqrt{a^2-x^2}\cos^{-1}\frac{x}{a}$$

**14.483** $$\int \tan^{-1}\frac{x}{a}\,dx = x\tan^{-1}\frac{x}{a} - \frac{a}{2}\ln(x^2+a^2)$$

**14.484** $$\int x\tan^{-1}\frac{x}{a}\,dx = \tfrac{1}{2}(x^2+a^2)\tan^{-1}\frac{x}{a} - \frac{ax}{2}$$

**14.485** $$\int x^2\tan^{-1}\frac{x}{a}\,dx = \frac{x^3}{3}\tan^{-1}\frac{x}{a} - \frac{ax^2}{6} + \frac{a^3}{6}\ln(x^2+a^2)$$

**14.486** $$\int \frac{\tan^{-1}(x/a)}{x}\,dx = \frac{x}{a} - \frac{(x/a)^3}{3^2} + \frac{(x/a)^5}{5^2} - \frac{(x/a)^7}{7^2} + \cdots$$

**14.487** $$\int \frac{\tan^{-1}(x/a)}{x^2}\,dx = -\frac{1}{x}\tan^{-1}\frac{x}{a} - \frac{1}{2a}\ln\left(\frac{x^2+a^2}{x^2}\right)$$

**14.488** $$\int \cot^{-1}\frac{x}{a}\,dx = x\cot^{-1}\frac{x}{a} + \frac{a}{2}\ln(x^2+a^2)$$

**14.489** $$\int x\cot^{-1}\frac{x}{a}\,dx = \tfrac{1}{2}(x^2+a^2)\cot^{-1}\frac{x}{a} + \frac{ax}{2}$$

**14.490** $$\int x^2\cot^{-1}\frac{x}{a}\,dx = \frac{x^3}{3}\cot^{-1}\frac{x}{a} + \frac{ax^2}{6} - \frac{a^3}{6}\ln(x^2+a^2)$$

**14.491** $$\int \frac{\cot^{-1}(x/a)}{x}\,dx = \frac{\pi}{2}\ln x - \int \frac{\tan^{-1}(x/a)}{x}\,dx$$ [See 14.486]

**14.492** $$\int \frac{\cot^{-1}(x/a)}{x^2}\,dx = -\frac{\cot^{-1}(x/a)}{x} + \frac{1}{2a}\ln\left(\frac{x^2+a^2}{x^2}\right)$$

**14.493** $$\int \sec^{-1}\frac{x}{a}\,dx = \begin{cases} x\sec^{-1}\dfrac{x}{a} - a\ln\left(x+\sqrt{x^2-a^2}\right) & 0 < \sec^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\[2ex] x\sec^{-1}\dfrac{x}{a} + a\ln\left(x+\sqrt{x^2-a^2}\right) & \dfrac{\pi}{2} < \sec^{-1}\dfrac{x}{a} < \pi \end{cases}$$

**14.494** $$\int x\sec^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^2}{2}\sec^{-1}\dfrac{x}{a} - \dfrac{a\sqrt{x^2-a^2}}{2} & 0 < \sec^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\[2ex] \dfrac{x^2}{2}\sec^{-1}\dfrac{x}{a} + \dfrac{a\sqrt{x^2-a^2}}{2} & \dfrac{\pi}{2} < \sec^{-1}\dfrac{x}{a} < \pi \end{cases}$$

**14.495** $$\int x^2\sec^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^3}{3}\sec^{-1}\dfrac{x}{a} - \dfrac{ax\sqrt{x^2-a^2}}{6} - \dfrac{a^3}{6}\ln\left(x+\sqrt{x^2-a^2}\right) & 0 < \sec^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\[2ex] \dfrac{x^3}{3}\sec^{-1}\dfrac{x}{a} + \dfrac{ax\sqrt{x^2-a^2}}{6} + \dfrac{a^3}{6}\ln\left(x+\sqrt{x^2-a^2}\right) & \dfrac{\pi}{2} < \sec^{-1}\dfrac{x}{a} < \pi \end{cases}$$

**14.496**
$$\int \frac{\sec^{-1}(x/a)}{x}\,dx = \frac{\pi}{2}\ln x + \frac{a}{x} + \frac{(a/x)^3}{2\cdot 3\cdot 3} + \frac{1\cdot 3(a/x)^5}{2\cdot 4\cdot 5\cdot 5} + \frac{1\cdot 3\cdot 5(a/x)^7}{2\cdot 4\cdot 6\cdot 7\cdot 7} + \cdots$$

**14.497**
$$\int \frac{\sec^{-1}(x/a)}{x^2}\,dx = \begin{cases} -\dfrac{\sec^{-1}(x/a)}{x} + \dfrac{\sqrt{x^2-a^2}}{ax} & 0 < \sec^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\[2ex] -\dfrac{\sec^{-1}(x/a)}{x} - \dfrac{\sqrt{x^2-a^2}}{ax} & \dfrac{\pi}{2} < \sec^{-1}\dfrac{x}{a} < \pi \end{cases}$$

**14.498**
$$\int \csc^{-1}\frac{x}{a}\,dx = \begin{cases} x\csc^{-1}\dfrac{x}{a} + a\ln\left(x + \sqrt{x^2-a^2}\right) & 0 < \csc^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\[2ex] x\csc^{-1}\dfrac{x}{a} - a\ln\left(x + \sqrt{x^2-a^2}\right) & -\dfrac{\pi}{2} < \csc^{-1}\dfrac{x}{a} < 0 \end{cases}$$

**14.499**
$$\int x\csc^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^2}{2}\csc^{-1}\dfrac{x}{a} + \dfrac{a\sqrt{x^2-a^2}}{2} & 0 < \csc^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\[2ex] \dfrac{x^2}{2}\csc^{-1}\dfrac{x}{a} - \dfrac{a\sqrt{x^2-a^2}}{2} & -\dfrac{\pi}{2} < \csc^{-1}\dfrac{x}{a} < 0 \end{cases}$$

**14.500**
$$\int x^2\csc^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^3}{3}\csc^{-1}\dfrac{x}{a} + \dfrac{ax\sqrt{x^2-a^2}}{6} + \dfrac{a^3}{6}\ln\left(x + \sqrt{x^2-a^2}\right) & 0 < \csc^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\[2ex] \dfrac{x^3}{3}\csc^{-1}\dfrac{x}{a} - \dfrac{ax\sqrt{x^2-a^2}}{6} - \dfrac{a^3}{6}\ln\left(x + \sqrt{x^2-a^2}\right) & -\dfrac{\pi}{2} < \csc^{-1}\dfrac{x}{a} < 0 \end{cases}$$

**14.501**
$$\int \frac{\csc^{-1}(x/a)}{x}\,dx = -\left(\frac{a}{x} + \frac{(a/x)^3}{2\cdot 3\cdot 3} + \frac{1\cdot 3(a/x)^5}{2\cdot 4\cdot 5\cdot 5} + \frac{1\cdot 3\cdot 5(a/x)^7}{2\cdot 4\cdot 6\cdot 7\cdot 7} + \cdots\right)$$

**14.502**
$$\int \frac{\csc^{-1}(x/a)}{x^2}\,dx = \begin{cases} -\dfrac{\csc^{-1}(x/a)}{x} - \dfrac{\sqrt{x^2-a^2}}{ax} & 0 < \csc^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\[2ex] -\dfrac{\csc^{-1}(x/a)}{x} + \dfrac{\sqrt{x^2-a^2}}{ax} & -\dfrac{\pi}{2} < \csc^{-1}\dfrac{x}{a} < 0 \end{cases}$$

**14.503**
$$\int x^m\sin^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\sin^{-1}\frac{x}{a} - \frac{1}{m+1}\int \frac{x^{m+1}}{\sqrt{a^2-x^2}}\,dx$$

**14.504**
$$\int x^m\cos^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\cos^{-1}\frac{x}{a} + \frac{1}{m+1}\int \frac{x^{m+1}}{\sqrt{a^2-x^2}}\,dx$$

**14.505**
$$\int x^m\tan^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\tan^{-1}\frac{x}{a} - \frac{a}{m+1}\int \frac{x^{m+1}}{x^2+a^2}\,dx$$

**14.506**
$$\int x^m\cot^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\cot^{-1}\frac{x}{a} + \frac{a}{m+1}\int \frac{x^{m+1}}{x^2+a^2}\,dx$$

**14.507**
$$\int x^m\sec^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^{m+1}\sec^{-1}(x/a)}{m+1} - \dfrac{a}{m+1}\displaystyle\int \frac{x^m\,dx}{\sqrt{x^2-a^2}} & 0 < \sec^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\[3ex] \dfrac{x^{m+1}\sec^{-1}(x/a)}{m+1} + \dfrac{a}{m+1}\displaystyle\int \frac{x^m\,dx}{\sqrt{x^2-a^2}} & \dfrac{\pi}{2} < \sec^{-1}\dfrac{x}{a} < \pi \end{cases}$$

**14.508**
$$\int x^m\csc^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^{m+1}\csc^{-1}(x/a)}{m+1} + \dfrac{a}{m+1}\displaystyle\int \frac{x^m\,dx}{\sqrt{x^2-a^2}} & 0 < \csc^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\[3ex] \dfrac{x^{m+1}\csc^{-1}(x/a)}{m+1} - \dfrac{a}{m+1}\displaystyle\int \frac{x^m\,dx}{\sqrt{x^2-a^2}} & -\dfrac{\pi}{2} < \csc^{-1}\dfrac{x}{a} < 0 \end{cases}$$

## INTEGRALS INVOLVING $e^{ax}$

**14.509** $$\int e^{ax}\,dx = \frac{e^{ax}}{a}$$

**14.510** $$\int xe^{ax}\,dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right)$$

**14.511** $$\int x^2e^{ax}\,dx = \frac{e^{ax}}{a}\left(x^2 - \frac{2x}{a} + \frac{2}{a^2}\right)$$

**14.512** $$\begin{aligned}\int x^ne^{ax}\,dx &= \frac{x^ne^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx \\ &= \frac{e^{ax}}{a}\left(x^n - \frac{nx^{n-1}}{a} + \frac{n(n-1)x^{n-2}}{a^2} - \cdots \frac{(-1)^n n!}{a^n}\right) \quad \text{if } n = \text{positive integer}\end{aligned}$$

**14.513** $$\int \frac{e^{ax}}{x}\,dx = \ln x + \frac{ax}{1\cdot 1!} + \frac{(ax)^2}{2\cdot 2!} + \frac{(ax)^3}{3\cdot 3!} + \cdots$$

**14.514** $$\int \frac{e^{ax}}{x^n}\,dx = \frac{-e^{ax}}{(n-1)x^{n-1}} + \frac{a}{n-1}\int \frac{e^{ax}}{x^{n-1}}\,dx$$

**14.515** $$\int \frac{dx}{p + qe^{ax}} = \frac{x}{p} - \frac{1}{ap}\ln\,(p + qe^{ax})$$

**14.516** $$\int \frac{dx}{(p + qe^{ax})^2} = \frac{x}{p^2} + \frac{1}{ap(p + qe^{ax})} - \frac{1}{ap^2}\ln\,(p + qe^{ax})$$

**14.517** $$\int \frac{dx}{pe^{ax} + qe^{-ax}} = \begin{cases} \dfrac{1}{a\sqrt{pq}}\tan^{-1}\left(\sqrt{\dfrac{p}{q}}\,e^{ax}\right) \\ \dfrac{1}{2a\sqrt{-pq}}\ln\left(\dfrac{e^{ax} - \sqrt{-q/p}}{e^{ax} + \sqrt{-q/p}}\right) \end{cases}$$

**14.518** $$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2 + b^2}$$

**14.519** $$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2}$$

**14.520** $$\int xe^{ax}\sin bx\,dx = \frac{xe^{ax}(a\sin bx - b\cos bx)}{a^2 + b^2} - \frac{e^{ax}\{(a^2 - b^2)\sin bx - 2ab\cos bx\}}{(a^2 + b^2)^2}$$

**14.521** $$\int xe^{ax}\cos bx\,dx = \frac{xe^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2} - \frac{e^{ax}\{(a^2 - b^2)\cos bx + 2ab\sin bx\}}{(a^2 + b^2)^2}$$

**14.522** $$\int e^{ax}\ln x\,dx = \frac{e^{ax}\ln x}{a} - \frac{1}{a}\int \frac{e^{ax}}{x}\,dx$$

**14.523** $$\int e^{ax}\sin^n bx\,dx = \frac{e^{ax}\sin^{n-1} bx}{a^2 + n^2b^2}(a\sin bx - nb\cos bx) + \frac{n(n-1)b^2}{a^2 + n^2b^2}\int e^{ax}\sin^{n-2} bx\,dx$$

**14.524** $$\int e^{ax}\cos^n bx\,dx = \frac{e^{ax}\cos^{n-1} bx}{a^2 + n^2b^2}(a\cos bx + nb\sin bx) + \frac{n(n-1)b^2}{a^2 + n^2b^2}\int e^{ax}\cos^{n-2} bx\,dx$$

## INTEGRALS INVOLVING $\ln x$

**14.525** $\displaystyle\int \ln x\,dx = x\ln x - x$

**14.526** $\displaystyle\int x\ln x\,dx = \frac{x^2}{2}(\ln x - \tfrac{1}{2})$

**14.527** $\displaystyle\int x^m \ln x\,dx = \frac{x^{m+1}}{m+1}\left(\ln x - \frac{1}{m+1}\right)$ [If $m = -1$ see 14.528.]

**14.528** $\displaystyle\int \frac{\ln x}{x}\,dx = \frac{1}{2}\ln^2 x$

**14.529** $\displaystyle\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x}$

**14.530** $\displaystyle\int \ln^2 x\,dx = x\ln^2 x - 2x\ln x + 2x$

**14.531** $\displaystyle\int \frac{\ln^n x\,dx}{x} = \frac{\ln^{n+1} x}{n+1}$ [If $n = -1$ see 14.532.]

**14.532** $\displaystyle\int \frac{dx}{x\ln x} = \ln(\ln x)$

**14.533** $\displaystyle\int \frac{dx}{\ln x} = \ln(\ln x) + \ln x + \frac{\ln^2 x}{2\cdot 2!} + \frac{\ln^3 x}{3\cdot 3!} + \cdots$

**14.534** $\displaystyle\int \frac{x^m\,dx}{\ln x} = \ln(\ln x) + (m+1)\ln x + \frac{(m+1)^2\ln^2 x}{2\cdot 2!} + \frac{(m+1)^3\ln^3 x}{3\cdot 3!} + \cdots$

**14.535** $\displaystyle\int \ln^n x\,dx = x\ln^n x - n\int \ln^{n-1} x\,dx$

**14.536** $\displaystyle\int x^m \ln^n x\,dx = \frac{x^{m+1}\ln^n x}{m+1} - \frac{n}{m+1}\int x^m \ln^{n-1} x\,dx$

If $m = -1$ see 14.531.

**14.537** $\displaystyle\int \ln(x^2+a^2)\,dx = x\ln(x^2+a^2) - 2x + 2a\tan^{-1}\frac{x}{a}$

**14.538** $\displaystyle\int \ln(x^2-a^2)\,dx = x\ln(x^2-a^2) - 2x + a\ln\left(\frac{x+a}{x-a}\right)$

**14.539** $\displaystyle\int x^m \ln(x^2 \pm a^2)\,dx = \frac{x^{m+1}\ln(x^2 \pm a^2)}{m+1} - \frac{2}{m+1}\int \frac{x^{m+2}}{x^2 \pm a^2}\,dx$

## INTEGRALS INVOLVING $\sinh ax$

**14.540** $\displaystyle\int \sinh ax\,dx = \frac{\cosh ax}{a}$

**14.541** $\displaystyle\int x\sinh ax\,dx = \frac{x\cosh ax}{a} - \frac{\sinh ax}{a^2}$

**14.542** $\displaystyle\int x^2\sinh ax\,dx = \left(\frac{x^2}{a} + \frac{2}{a^3}\right)\cosh ax - \frac{2x}{a^2}\sinh ax$

**14.543** $$\int \frac{\sinh ax}{x}\,dx = ax + \frac{(ax)^3}{3\cdot 3!} + \frac{(ax)^5}{5\cdot 5!} + \cdots$$

**14.544** $$\int \frac{\sinh ax}{x^2}\,dx = -\frac{\sinh ax}{x} + a\int \frac{\cosh ax}{x}\,dx \qquad \text{[See 14.565]}$$

**14.545** $$\int \frac{dx}{\sinh ax} = \frac{1}{a}\ln\tanh\frac{ax}{2}$$

**14.546** $$\int \frac{x\,dx}{\sinh ax} = \frac{1}{a^2}\left\{ax - \frac{(ax)^3}{18} + \frac{7(ax)^5}{1800} - \cdots + \frac{2(-1)^n(2^{2n}-1)B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$$

**14.547** $$\int \sinh^2 ax\,dx = \frac{\sinh ax\cosh ax}{2a} - \frac{x}{2}$$

**14.548** $$\int x\sinh^2 ax\,dx = \frac{x\sinh 2ax}{4a} - \frac{\cosh 2ax}{8a^2} - \frac{x^2}{4}$$

**14.549** $$\int \frac{dx}{\sinh^2 ax} = -\frac{\coth ax}{a}$$

**14.550** $$\int \sinh ax\sinh px\,dx = \frac{\sinh(a+p)x}{2(a+p)} - \frac{\sinh(a-p)x}{2(a-p)}$$

For $a = \pm p$ see 14.547.

**14.551** $$\int \sinh ax\sin px\,dx = \frac{a\cosh ax\sin px - p\sinh ax\cos px}{a^2+p^2}$$

**14.552** $$\int \sinh ax\cos px\,dx = \frac{a\cosh ax\cos px + p\sinh ax\sin px}{a^2+p^2}$$

**14.553** $$\int \frac{dx}{p+q\sinh ax} = \frac{1}{a\sqrt{p^2+q^2}}\ln\left(\frac{qe^{ax}+p-\sqrt{p^2+q^2}}{qe^{ax}+p+\sqrt{p^2+q^2}}\right)$$

**14.554** $$\int \frac{dx}{(p+q\sinh ax)^2} = \frac{-q\cosh ax}{a(p^2+q^2)(p+q\sinh ax)} + \frac{p}{p^2+q^2}\int \frac{dx}{p+q\sinh ax}$$

**14.555** $$\int \frac{dx}{p^2+q^2\sinh^2 ax} = \begin{cases} \dfrac{1}{ap\sqrt{q^2-p^2}}\tan^{-1}\dfrac{\sqrt{q^2-p^2}\tanh ax}{p} \\[2ex] \dfrac{1}{2ap\sqrt{p^2-q^2}}\ln\left(\dfrac{p+\sqrt{p^2-q^2}\tanh ax}{p-\sqrt{p^2-q^2}\tanh ax}\right) \end{cases}$$

**14.556** $$\int \frac{dx}{p^2-q^2\sinh^2 ax} = \frac{1}{2ap\sqrt{p^2+q^2}}\ln\left(\frac{p+\sqrt{p^2+q^2}\tanh ax}{p-\sqrt{p^2+q^2}\tanh ax}\right)$$

**14.557** $$\int x^m\sinh ax\,dx = \frac{x^m\cosh ax}{a} - \frac{m}{a}\int x^{m-1}\cosh ax\,dx \qquad \text{[See 14.585]}$$

**14.558** $$\int \sinh^n ax\,dx = \frac{\sinh^{n-1}ax\cosh ax}{an} - \frac{n-1}{n}\int \sinh^{n-2}ax\,dx$$

**14.559** $$\int \frac{\sinh ax}{x^n}\,dx = \frac{-\sinh ax}{(n-1)x^{n-1}} + \frac{a}{n-1}\int \frac{\cosh ax}{x^{n-1}}\,dx \qquad \text{[See 14.587]}$$

**14.560** $$\int \frac{dx}{\sinh^n ax} = \frac{-\cosh ax}{a(n-1)\sinh^{n-1}ax} - \frac{n-2}{n-1}\int \frac{dx}{\sinh^{n-2}ax}$$

**14.561** $$\int \frac{x\,dx}{\sinh^n ax} = \frac{-x\cosh ax}{a(n-1)\sinh^{n-1}ax} - \frac{1}{a^2(n-1)(n-2)\sinh^{n-2}ax} - \frac{n-2}{n-1}\int \frac{x\,dx}{\sinh^{n-2}ax}$$

## INTEGRALS INVOLVING cosh $ax$

**14.562** $\int \cosh ax\,dx = \frac{\sinh ax}{a}$

**14.563** $\int x\cosh ax\,dx = \frac{x\sinh ax}{a} - \frac{\cosh ax}{a^2}$

**14.564** $\int x^2\cosh ax\,dx = -\frac{2x\cosh ax}{a^2} + \left(\frac{x^2}{a} + \frac{2}{a^3}\right)\sinh ax$

**14.565** $\int \frac{\cosh ax}{x}\,dx = \ln x + \frac{(ax)^2}{2\cdot 2!} + \frac{(ax)^4}{4\cdot 4!} + \frac{(ax)^6}{6\cdot 6!} + \cdots$

**14.566** $\int \frac{\cosh ax}{x^2}\,dx = -\frac{\cosh ax}{x} + a\int \frac{\sinh ax}{x}\,dx$ [See 14.543]

**14.567** $\int \frac{dx}{\cosh ax} = \frac{2}{a}\tan^{-1} e^{ax}$

**14.568** $\int \frac{x\,dx}{\cosh ax} = \frac{1}{a^2}\left\{\frac{(ax)^2}{2} - \frac{(ax)^4}{8} + \frac{5(ax)^6}{144} + \cdots + \frac{(-1)^n E_n (ax)^{2n+2}}{(2n+2)(2n)!} + \cdots\right\}$

**14.569** $\int \cosh^2 ax\,dx = \frac{x}{2} + \frac{\sinh ax\cosh ax}{2a}$

**14.570** $\int x\cosh^2 ax\,dx = \frac{x^2}{4} + \frac{x\sinh 2ax}{4a} - \frac{\cosh 2ax}{8a^2}$

**14.571** $\int \frac{dx}{\cosh^2 ax} = \frac{\tanh ax}{a}$

**14.572** $\int \cosh ax\cosh px\,dx = \frac{\sinh(a-p)x}{2(a-p)} + \frac{\sinh(a+p)x}{2(a+p)}$

**14.573** $\int \cosh ax\sin px\,dx = \frac{a\sinh ax\sin px - p\cosh ax\cos px}{a^2+p^2}$

**14.574** $\int \cosh ax\cos px\,dx = \frac{a\sinh ax\cos px + p\cosh ax\sin px}{a^2+p^2}$

**14.575** $\int \frac{dx}{\cosh ax + 1} = \frac{1}{a}\tanh\frac{ax}{2}$

**14.576** $\int \frac{dx}{\cosh ax - 1} = -\frac{1}{a}\coth\frac{ax}{2}$

**14.577** $\int \frac{x\,dx}{\cosh ax + 1} = \frac{x}{a}\tanh\frac{ax}{2} - \frac{2}{a^2}\ln\cosh\frac{ax}{2}$

**14.578** $\int \frac{x\,dx}{\cosh ax - 1} = -\frac{x}{a}\coth\frac{ax}{2} + \frac{2}{a^2}\ln\sinh\frac{ax}{2}$

**14.579** $\int \frac{dx}{(\cosh ax + 1)^2} = \frac{1}{2a}\tanh\frac{ax}{2} - \frac{1}{6a}\tanh^3\frac{ax}{2}$

**14.580** $\int \frac{dx}{(\cosh ax - 1)^2} = \frac{1}{2a}\coth\frac{ax}{2} - \frac{1}{6a}\coth^3\frac{ax}{2}$

**14.581** $\int \frac{dx}{p + q\cosh ax} = \begin{cases} \dfrac{2}{a\sqrt{q^2-p^2}}\tan^{-1}\dfrac{qe^{ax}+p}{\sqrt{q^2-p^2}} \\ \dfrac{1}{a\sqrt{p^2-q^2}}\ln\left(\dfrac{qe^{ax}+p-\sqrt{p^2-q^2}}{qe^{ax}+p+\sqrt{p^2-q^2}}\right) \end{cases}$

**14.582** $\int \frac{dx}{(p + q\cosh ax)^2} = \frac{q\sinh ax}{a(q^2-p^2)(p+q\cosh ax)} - \frac{p}{q^2-p^2}\int \frac{dx}{p+q\cosh ax}$

**14.583** $$\int \frac{dx}{p^2 - q^2 \cosh^2 ax} = \begin{cases} \dfrac{1}{2ap\sqrt{p^2-q^2}} \ln \left( \dfrac{p \tanh ax + \sqrt{p^2-q^2}}{p \tanh ax - \sqrt{p^2-q^2}} \right) \\ \dfrac{-1}{ap\sqrt{q^2-p^2}} \tan^{-1} \dfrac{p \tanh ax}{\sqrt{q^2-p^2}} \end{cases}$$

**14.584** $$\int \frac{dx}{p^2 + q^2 \cosh^2 ax} = \begin{cases} \dfrac{1}{2ap\sqrt{p^2+q^2}} \ln \left( \dfrac{p \tanh ax + \sqrt{p^2+q^2}}{p \tanh ax - \sqrt{p^2+q^2}} \right) \\ \dfrac{1}{ap\sqrt{p^2+q^2}} \tan^{-1} \dfrac{p \tanh ax}{\sqrt{p^2+q^2}} \end{cases}$$

**14.585** $$\int x^m \cosh ax \, dx = \frac{x^m \sinh ax}{a} - \frac{m}{a} \int x^{m-1} \sinh ax \, dx$$ [See 14.557]

**14.586** $$\int \cosh^n ax \, dx = \frac{\cosh^{n-1} ax \sinh ax}{an} + \frac{n-1}{n} \int \cosh^{n-2} ax \, dx$$

**14.587** $$\int \frac{\cosh ax}{x^n} \, dx = \frac{-\cosh ax}{(n-1)x^{n-1}} + \frac{a}{n-1} \int \frac{\sinh ax}{x^{n-1}} \, dx$$ [See 14.559]

**14.588** $$\int \frac{dx}{\cosh^n ax} = \frac{\sinh ax}{a(n-1) \cosh^{n-1} ax} + \frac{n-2}{n-1} \int \frac{dx}{\cosh^{n-2} ax}$$

**14.589** $$\int \frac{x \, dx}{\cosh^n ax} = \frac{x \sinh ax}{a(n-1) \cosh^{n-1} ax} + \frac{1}{(n-1)(n-2)a^2 \cosh^{n-2} ax} + \frac{n-2}{n-1} \int \frac{x \, dx}{\cosh^{n-2} ax}$$

## INTEGRALS INVOLVING $\sinh ax$ AND $\cosh ax$

**14.590** $$\int \sinh ax \cosh ax \, dx = \frac{\sinh^2 ax}{2a}$$

**14.591** $$\int \sinh px \cosh qx \, dx = \frac{\cosh (p+q)x}{2(p+q)} + \frac{\cosh (p-q)x}{2(p-q)}$$

**14.592** $$\int \sinh^n ax \cosh ax \, dx = \frac{\sinh^{n+1} ax}{(n+1)a}$$ [If $n = -1$, see 14.615.]

**14.593** $$\int \cosh^n ax \sinh ax \, dx = \frac{\cosh^{n+1} ax}{(n+1)a}$$ [If $n = -1$, see 14.604.]

**14.594** $$\int \sinh^2 ax \cosh^2 ax \, dx = \frac{\sinh 4ax}{32a} - \frac{x}{8}$$

**14.595** $$\int \frac{dx}{\sinh ax \cosh ax} = \frac{1}{a} \ln \tanh ax$$

**14.596** $$\int \frac{dx}{\sinh^2 ax \cosh ax} = -\frac{1}{a} \tan^{-1} \sinh ax - \frac{\operatorname{csch} ax}{a}$$

**14.597** $$\int \frac{dx}{\sinh ax \cosh^2 ax} = \frac{\operatorname{sech} ax}{a} + \frac{1}{a} \ln \tanh \frac{ax}{2}$$

**14.598** $$\int \frac{dx}{\sinh^2 ax \cosh^2 ax} = -\frac{2 \coth 2ax}{a}$$

**14.599** $$\int \frac{\sinh^2 ax}{\cosh ax} \, dx = \frac{\sinh ax}{a} - \frac{1}{a} \tan^{-1} \sinh ax$$

**14.600** $$\int \frac{\cosh^2 ax}{\sinh ax} \, dx = \frac{\cosh ax}{a} + \frac{1}{a} \ln \tanh \frac{ax}{2}$$

**14.601** $$\int \frac{dx}{\cosh ax \, (1 + \sinh ax)} = \frac{1}{2a} \ln \left( \frac{1 + \sinh ax}{\cosh ax} \right) + \frac{1}{a} \tan^{-1} e^{ax}$$

**14.602** $$\int \frac{dx}{\sinh ax\,(\cosh ax+1)} = \frac{1}{2a}\ln\tanh\frac{ax}{2} + \frac{1}{2a(\cosh ax+1)}$$

**14.603** $$\int \frac{dx}{\sinh ax\,(\cosh ax-1)} = -\frac{1}{2a}\ln\tanh\frac{ax}{2} - \frac{1}{2a(\cosh ax-1)}$$

## INTEGRALS INVOLVING tanh $ax$

**14.604** $$\int \tanh ax\,dx = \frac{1}{a}\ln\cosh ax$$

**14.605** $$\int \tanh^2 ax\,dx = x - \frac{\tanh ax}{a}$$

**14.606** $$\int \tanh^3 ax\,dx = \frac{1}{a}\ln\cosh ax - \frac{\tanh^2 ax}{2a}$$

**14.607** $$\int \tanh^n ax\,\operatorname{sech}^2 ax\,dx = \frac{\tanh^{n+1} ax}{(n+1)a}$$

**14.608** $$\int \frac{\operatorname{sech}^2 ax}{\tanh ax}\,dx = \frac{1}{a}\ln\tanh ax$$

**14.609** $$\int \frac{dx}{\tanh ax} = \frac{1}{a}\ln\sinh ax$$

**14.610** $$\int x\tanh ax\,dx = \frac{1}{a^2}\left\{\frac{(ax)^3}{3} - \frac{(ax)^5}{15} + \frac{2(ax)^7}{105} - \cdots \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$$

**14.611** $$\int x\tanh^2 ax\,dx = \frac{x^2}{2} - \frac{x\tanh ax}{a} + \frac{1}{a^2}\ln\cosh ax$$

**14.612** $$\int \frac{\tanh ax}{x}\,dx = ax - \frac{(ax)^3}{9} + \frac{2(ax)^5}{75} - \cdots \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_n(ax)^{2n-1}}{(2n-1)(2n)!} + \cdots$$

**14.613** $$\int \frac{dx}{p+q\tanh ax} = \frac{px}{p^2-q^2} - \frac{q}{a(p^2-q^2)}\ln(q\sinh ax + p\cosh ax)$$

**14.614** $$\int \tanh^n ax\,dx = \frac{-\tanh^{n-1} ax}{a(n-1)} + \int \tanh^{n-2} ax\,dx$$

## INTEGRALS INVOLVING coth $ax$

**14.615** $$\int \coth ax\,dx = \frac{1}{a}\ln\sinh ax$$

**14.616** $$\int \coth^2 ax\,dx = x - \frac{\coth ax}{a}$$

**14.617** $$\int \coth^3 ax\,dx = \frac{1}{a}\ln\sinh ax - \frac{\coth^2 ax}{2a}$$

**14.618** $$\int \coth^n ax\,\operatorname{csch}^2 ax\,dx = -\frac{\coth^{n+1} ax}{(n+1)a}$$

**14.619** $$\int \frac{\operatorname{csch}^2 ax}{\coth ax}\,dx = -\frac{1}{a}\ln\coth ax$$

**14.620** $$\int \frac{dx}{\coth ax} = \frac{1}{a}\ln\cosh ax$$

**14.621** $\int x \coth ax \, dx = \frac{1}{a^2}\left\{ax + \frac{(ax)^3}{9} - \frac{(ax)^5}{225} + \cdots \frac{(-1)^{n-1}2^{2n}B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$

**14.622** $\int x \coth^2 ax \, dx = \frac{x^2}{2} - \frac{x \coth ax}{a} + \frac{1}{a^2} \ln \sinh ax$

**14.623** $\int \frac{\coth ax}{x} \, dx = -\frac{1}{ax} + \frac{ax}{3} - \frac{(ax)^3}{135} + \cdots \frac{(-1)^n 2^{2n} B_n (ax)^{2n-1}}{(2n-1)(2n)!} + \cdots$

**14.624** $\int \frac{dx}{p + q \coth ax} = \frac{px}{p^2 - q^2} - \frac{q}{a(p^2 - q^2)} \ln (p \sinh ax + q \cosh ax)$

**14.625** $\int \coth^n ax \, dx = -\frac{\coth^{n-1} ax}{a(n-1)} + \int \coth^{n-2} ax \, dx$

## INTEGRALS INVOLVING sech $ax$

**14.626** $\int \operatorname{sech} ax \, dx = \frac{2}{a} \tan^{-1} e^{ax}$

**14.627** $\int \operatorname{sech}^2 ax \, dx = \frac{\tanh ax}{a}$

**14.628** $\int \operatorname{sech}^3 ax \, dx = \frac{\operatorname{sech} ax \tanh ax}{2a} + \frac{1}{2a} \tan^{-1} \sinh ax$

**14.629** $\int \operatorname{sech}^n ax \tanh ax \, dx = -\frac{\operatorname{sech}^n ax}{na}$

**14.630** $\int \frac{dx}{\operatorname{sech} ax} = \frac{\sinh ax}{a}$

**14.631** $\int x \operatorname{sech} ax \, dx = \frac{1}{a^2}\left\{\frac{(ax)^2}{2} - \frac{(ax)^4}{8} + \frac{5(ax)^6}{144} + \cdots \frac{(-1)^n E_n (ax)^{2n+2}}{(2n+2)(2n)!} + \cdots\right\}$

**14.632** $\int x \operatorname{sech}^2 ax \, dx = \frac{x \tanh ax}{a} - \frac{1}{a^2} \ln \cosh ax$

**14.633** $\int \frac{\operatorname{sech} ax}{x} \, dx = \ln x - \frac{(ax)^2}{4} + \frac{5(ax)^4}{96} - \frac{61(ax)^6}{4320} + \cdots \frac{(-1)^n E_n (ax)^{2n}}{2n(2n)!} + \cdots$

**14.634** $\int \frac{dx}{q + p \operatorname{sech} ax} = \frac{x}{q} - \frac{p}{q} \int \frac{dx}{p + q \cosh ax}$ [See 14.581]

**14.635** $\int \operatorname{sech}^n ax \, dx = \frac{\operatorname{sech}^{n-2} ax \tanh ax}{a(n-1)} + \frac{n-2}{n-1} \int \operatorname{sech}^{n-2} ax \, dx$

## INTEGRALS INVOLVING csch $ax$

**14.636** $\int \operatorname{csch} ax \, dx = \frac{1}{a} \ln \tanh \frac{ax}{2}$

**14.637** $\int \operatorname{csch}^2 ax \, dx = -\frac{\coth ax}{a}$

**14.638** $\int \operatorname{csch}^3 ax \, dx = -\frac{\operatorname{csch} ax \coth ax}{2a} - \frac{1}{2a} \ln \tanh \frac{ax}{2}$

**14.639** $\int \operatorname{csch}^n ax \coth ax \, dx = -\frac{\operatorname{csch}^n ax}{na}$

**14.640** $$\int \frac{dx}{\operatorname{csch} ax} = \frac{1}{a}\cosh ax$$

**14.641** $$\int x \operatorname{csch} ax\, dx = \frac{1}{a^2}\left\{ax - \frac{(ax)^3}{18} + \frac{7(ax)^5}{1800} + \cdots + \frac{2(-1)^n(2^{2n-1}-1)B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$$

**14.642** $$\int x \operatorname{csch}^2 ax\, dx = -\frac{x \coth ax}{a} + \frac{1}{a^2}\ln \sinh ax$$

**14.643** $$\int \frac{\operatorname{csch} ax}{x}\, dx = -\frac{1}{ax} - \frac{ax}{6} + \frac{7(ax)^3}{1080} + \cdots \frac{(-1)^n 2(2^{2n-1}-1)B_n(ax)^{2n-1}}{(2n-1)(2n)!} + \cdots$$

**14.644** $$\int \frac{dx}{q + p \operatorname{csch} ax} = \frac{x}{q} - \frac{p}{q}\int \frac{dx}{p + q \sinh ax} \qquad \text{[See 14.553]}$$

**14.645** $$\int \operatorname{csch}^n ax\, dx = \frac{-\operatorname{csch}^{n-2} ax \coth ax}{a(n-1)} - \frac{n-2}{n-1}\int \operatorname{csch}^{n-2} ax\, dx$$

## INTEGRALS INVOLVING INVERSE HYPERBOLIC FUNCTIONS

**14.646** $$\int \sinh^{-1}\frac{x}{a}\, dx = x \sinh^{-1}\frac{x}{a} - \sqrt{x^2+a^2}$$

**14.647** $$\int x \sinh^{-1}\frac{x}{a}\, dx = \left(\frac{x^2}{2} + \frac{a^2}{4}\right)\sinh^{-1}\frac{x}{a} - \frac{x\sqrt{x^2+a^2}}{4}$$

**14.648** $$\int x^2 \sinh^{-1}\frac{x}{a}\, dx = \frac{x^3}{3}\sinh^{-1}\frac{x}{a} + \frac{(2a^2-x^2)\sqrt{x^2+a^2}}{9}$$

**14.649** $$\int \frac{\sinh^{-1}(x/a)}{x}\, dx = \begin{cases} \dfrac{x}{a} - \dfrac{(x/a)^3}{2\cdot 3\cdot 3} + \dfrac{1\cdot 3(x/a)^5}{2\cdot 4\cdot 5\cdot 5} - \dfrac{1\cdot 3\cdot 5(x/a)^7}{2\cdot 4\cdot 6\cdot 7\cdot 7} + \cdots & |x| < a \\[2ex] \dfrac{\ln^2(2x/a)}{2} - \dfrac{(a/x)^2}{2\cdot 2\cdot 2} + \dfrac{1\cdot 3(a/x)^4}{2\cdot 4\cdot 4\cdot 4} - \dfrac{1\cdot 3\cdot 5(a/x)^6}{2\cdot 4\cdot 6\cdot 6\cdot 6} + \cdots & x > a \\[2ex] -\dfrac{\ln^2(-2x/a)}{2} + \dfrac{(a/x)^2}{2\cdot 2\cdot 2} - \dfrac{1\cdot 3(a/x)^4}{2\cdot 4\cdot 4\cdot 4} + \dfrac{1\cdot 3\cdot 5(a/x)^6}{2\cdot 4\cdot 6\cdot 6\cdot 6} - \cdots & x < -a \end{cases}$$

**14.650** $$\int \frac{\sinh^{-1}(x/a)}{x^2}\, dx = -\frac{\sinh^{-1}(x/a)}{x} - \frac{1}{a}\ln\left(\frac{a + \sqrt{x^2+a^2}}{x}\right)$$

**14.651** $$\int \cosh^{-1}\frac{x}{a}\, dx = \begin{cases} x\cosh^{-1}(x/a) - \sqrt{x^2-a^2}, & \cosh^{-1}(x/a) > 0 \\ x\cosh^{-1}(x/a) + \sqrt{x^2-a^2}, & \cosh^{-1}(x/a) < 0 \end{cases}$$

**14.652** $$\int x \cosh^{-1}\frac{x}{a}\, dx = \begin{cases} \frac{1}{4}(2x^2-a^2)\cosh^{-1}(x/a) - \frac{1}{4}x\sqrt{x^2-a^2}, & \cosh^{-1}(x/a) > 0 \\ \frac{1}{4}(2x^2-a^2)\cosh^{-1}(x/a) + \frac{1}{4}x\sqrt{x^2-a^2}, & \cosh^{-1}(x/a) < 0 \end{cases}$$

**14.653** $$\int x^2 \cosh^{-1}\frac{x}{a}\, dx = \begin{cases} \frac{1}{3}x^3\cosh^{-1}(x/a) - \frac{1}{9}(x^2+2a^2)\sqrt{x^2-a^2}, & \cosh^{-1}(x/a) > 0 \\ \frac{1}{8}x^3\cosh^{-1}(x/a) + \frac{1}{9}(x^2+2a^2)\sqrt{x^2-a^2}, & \cosh^{-1}(x/a) < 0 \end{cases}$$

**14.654** $$\int \frac{\cosh^{-1}(x/a)}{x}\, dx = \pm\left[\frac{1}{2}\ln^2(2x/a) + \frac{(a/x)^2}{2\cdot 2\cdot 2} + \frac{1\cdot 3(a/x)^4}{2\cdot 4\cdot 4\cdot 4} + \frac{1\cdot 3\cdot 5(a/x)^6}{2\cdot 4\cdot 6\cdot 6\cdot 6} + \cdots\right]$$

$+$ if $\cosh^{-1}(x/a) > 0$, $-$ if $\cosh^{-1}(x/a) < 0$

**14.655** $$\int \frac{\cosh^{-1}(x/a)}{x^2}\, dx = -\frac{\cosh^{-1}(x/a)}{x} \mp \frac{1}{a}\ln\left(\frac{a + \sqrt{x^2+a^2}}{x}\right)$$ [$-$ if $\cosh^{-1}(x/a) > 0$, $+$ if $\cosh^{-1}(x/a) < 0$]

**14.656** $$\int \tanh^{-1}\frac{x}{a}\, dx = x\tanh^{-1}\frac{x}{a} + \frac{a}{2}\ln(a^2-x^2)$$

**14.657** $$\int x \tanh^{-1}\frac{x}{a}\, dx = \frac{ax}{2} + \frac{1}{2}(x^2-a^2)\tanh^{-1}\frac{x}{a}$$

**14.658** $$\int x^2 \tanh^{-1}\frac{x}{a}\, dx = \frac{ax^2}{6} + \frac{x^3}{3}\tanh^{-1}\frac{x}{a} + \frac{a^3}{6}\ln(a^2-x^2)$$

**14.659**
$$\int \frac{\tanh^{-1}(x/a)}{x}\,dx = \frac{x}{a} + \frac{(x/a)^3}{3^2} + \frac{(x/a)^5}{5^2} + \cdots$$

**14.660**
$$\int \frac{\tanh^{-1}(x/a)}{x^2}\,dx = -\frac{\tanh^{-1}(x/a)}{x} + \frac{1}{2a}\ln\left(\frac{x^2}{a^2-x^2}\right)$$

**14.661**
$$\int \coth^{-1}\frac{x}{a}\,dx = x\coth^{-1}x + \frac{a}{2}\ln(x^2-a^2)$$

**14.662**
$$\int x\coth^{-1}\frac{x}{a}\,dx = \frac{ax}{2} + \tfrac{1}{2}(x^2-a^2)\coth^{-1}\frac{x}{a}$$

**14.663**
$$\int x^2\coth^{-1}\frac{x}{a}\,dx = \frac{ax^2}{6} + \frac{x^3}{3}\coth^{-1}\frac{x}{a} + \frac{a^3}{6}\ln(x^2-a^2)$$

**14.664**
$$\int \frac{\coth^{-1}(x/a)}{x}\,dx = -\left(\frac{a}{x} + \frac{(a/x)^3}{3^2} + \frac{(a/x)^5}{5^2} + \cdots\right)$$

**14.665**
$$\int \frac{\coth^{-1}(x/a)}{x^2}\,dx = -\frac{\coth^{-1}(x/a)}{x} + \frac{1}{2a}\ln\left(\frac{x^2}{x^2-a^2}\right)$$

**14.666**
$$\int \operatorname{sech}^{-1}\frac{x}{a}\,dx = \begin{cases} x\operatorname{sech}^{-1}(x/a) + a\sin^{-1}(x/a), & \operatorname{sech}^{-1}(x/a) > 0 \\ x\operatorname{sech}^{-1}(x/a) - a\sin^{-1}(x/a), & \operatorname{sech}^{-1}(x/a) < 0 \end{cases}$$

**14.667**
$$\int x\operatorname{sech}^{-1}\frac{x}{a}\,dx = \begin{cases} \tfrac{1}{2}x^2\operatorname{sech}^{-1}(x/a) - \tfrac{1}{2}a\sqrt{a^2-x^2}, & \operatorname{sech}^{-1}(x/a) > 0 \\ \tfrac{1}{2}x^2\operatorname{sech}^{-1}(x/a) + \tfrac{1}{2}a\sqrt{a^2-x^2}, & \operatorname{sech}^{-1}(x/a) < 0 \end{cases}$$

**14.668**
$$\int \frac{\operatorname{sech}^{-1}(x/a)}{x}\,dx = \begin{cases} -\tfrac{1}{2}\ln(a/x)\ln(4a/x) - \dfrac{(x/a)^2}{2\cdot2\cdot2} - \dfrac{1\cdot3(x/a)^4}{2\cdot4\cdot4\cdot4} - \cdots, & \operatorname{sech}^{-1}(x/a) > 0 \\ \tfrac{1}{2}\ln(a/x)\ln(4a/x) + \dfrac{(x/a)^2}{2\cdot2\cdot2} + \dfrac{1\cdot3(x/a)^4}{2\cdot4\cdot4\cdot4} + \cdots, & \operatorname{sech}^{-1}(x/a) < 0 \end{cases}$$

**14.669**
$$\int \operatorname{csch}^{-1}\frac{x}{a}\,dx = x\operatorname{csch}^{-1}\frac{x}{a} \pm a\sinh^{-1}\frac{x}{a} \qquad [+ \text{ if } x>0,\ - \text{ if } x<0]$$

**14.670**
$$\int x\operatorname{csch}^{-1}\frac{x}{a}\,dx = \frac{x^2}{2}\operatorname{csch}^{-1}\frac{x}{a} \pm \frac{a\sqrt{x^2+a^2}}{2} \qquad [+ \text{ if } x>0,\ - \text{ if } x<0]$$

**14.671**
$$\int \frac{\operatorname{csch}^{-1}(x/a)}{x}\,dx = \begin{cases} \tfrac{1}{2}\ln(x/a)\ln(4a/x) + \dfrac{1(x/a)^2}{2\cdot2\cdot2} - \dfrac{1\cdot3(x/a)^4}{2\cdot4\cdot4\cdot4} + \cdots & 0<x<a \\ \tfrac{1}{2}\ln(-x/a)\ln(-x/4a) - \dfrac{(x/a)^2}{2\cdot2\cdot2} + \dfrac{1\cdot3(x/a)^4}{2\cdot4\cdot4\cdot4} - \cdots & -a<x<0 \\ -\dfrac{a}{x} + \dfrac{(a/x)^3}{2\cdot3\cdot3} - \dfrac{1\cdot3(a/x)^5}{2\cdot4\cdot5\cdot5} + \cdots & |x|>a \end{cases}$$

**14.672**
$$\int x^m\sinh^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\sinh^{-1}\frac{x}{a} - \frac{1}{m+1}\int\frac{x^{m+1}}{\sqrt{x^2+a^2}}\,dx$$

**14.673**
$$\int x^m\cosh^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^{m+1}}{m+1}\cosh^{-1}\dfrac{x}{a} - \dfrac{1}{m+1}\displaystyle\int\frac{x^{m+1}}{\sqrt{x^2-a^2}}\,dx & \cosh^{-1}(x/a) > 0 \\ \dfrac{x^{m+1}}{m+1}\cosh^{-1}\dfrac{x}{a} + \dfrac{1}{m+1}\displaystyle\int\frac{x^{m+1}}{\sqrt{x^2-a^2}}\,dx & \cosh^{-1}(x/a) < 0 \end{cases}$$

**14.674**
$$\int x^m\tanh^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\tanh^{-1}\frac{x}{a} - \frac{a}{m+1}\int\frac{x^{m+1}}{a^2-x^2}\,dx$$

**14.675**
$$\int x^m\coth^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\coth^{-1}\frac{x}{a} - \frac{a}{m+1}\int\frac{x^{m+1}}{a^2-x^2}\,dx$$

**14.676**
$$\int x^m\operatorname{sech}^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^{m+1}}{m+1}\operatorname{sech}^{-1}\dfrac{x}{a} + \dfrac{a}{m+1}\displaystyle\int\frac{x^m\,dx}{\sqrt{a^2-x^2}} & \operatorname{sech}^{-1}(x/a) > 0 \\ \dfrac{x^{m+1}}{m+1}\operatorname{sech}^{-1}\dfrac{x}{a} - \dfrac{a}{m+1}\displaystyle\int\frac{x^m\,dx}{\sqrt{a^2-x^2}} & \operatorname{sech}^{-1}(x/a) < 0 \end{cases}$$

**14.677**
$$\int x^m\operatorname{csch}^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\operatorname{csch}^{-1}\frac{x}{a} \pm \frac{a}{m+1}\int\frac{x^m\,dx}{\sqrt{x^2+a^2}} \qquad [+ \text{ if } x>0,\ - \text{ if } x<0]$$

# 15 DEFINITE INTEGRALS

## DEFINITION OF A DEFINITE INTEGRAL

Let $f(x)$ be defined in an interval $a \leqq x \leqq b$. Divide the interval into $n$ equal parts of length $\Delta x = (b-a)/n$. Then the definite integral of $f(x)$ between $x = a$ and $x = b$ is defined as

**15.1** $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \{f(a)\,\Delta x + f(a+\Delta x)\,\Delta x + f(a+2\Delta x)\,\Delta x + \cdots + f(a+(n-1)\,\Delta x)\,\Delta x\}$$

The limit will certainly exist if $f(x)$ is piecewise continuous.

If $f(x) = \dfrac{d}{dx}g(x)$, then by the fundamental theorem of the integral calculus the above definite integral can be evaluated by using the result

**15.2** $$\int_a^b f(x)\,dx = \int_a^b \frac{d}{dx}g(x)\,dx = g(x)\Big|_a^b = g(b) - g(a)$$

If the interval is infinite or if $f(x)$ has a singularity at some point in the interval, the definite integral is called an *improper integral* and can be defined by using appropriate limiting procedures. For example,

**15.3** $$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

**15.4** $$\int_{-\infty}^\infty f(x)\,dx = \lim_{\substack{a\to-\infty\\ b\to\infty}}\int_a^b f(x)\,dx$$

**15.5** $$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0}\int_a^{b-\epsilon} f(x)\,dx \qquad \text{if } b \text{ is a singular point}$$

**15.6** $$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0}\int_{a+\epsilon}^{b} f(x)\,dx \qquad \text{if } a \text{ is a singular point}$$

## GENERAL FORMULAS INVOLVING DEFINITE INTEGRALS

**15.7** $$\int_a^b \{f(x) \pm g(x) \pm h(x) \pm \cdots\}\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx \pm \int_a^b h(x)\,dx \pm \cdots$$

**15.8** $$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx \qquad \text{where } c \text{ is any constant}$$

**15.9** $$\int_a^a f(x)\,dx = 0$$

**15.10** $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

**15.11** $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

**15.12** $$\int_a^b f(x)\,dx = (b-a)\,f(c) \qquad \text{where } c \text{ is between } a \text{ and } b$$

This is called the *mean value theorem* for definite integrals and is valid if $f(x)$ is continuous in $a \leqq x \leqq b$.

**15.13** $$\int_a^b f(x)\,g(x)\,dx \;=\; f(c)\int_a^b g(x)\,dx \qquad \text{where } c \text{ is between } a \text{ and } b$$

This is a generalization of 15.12 and is valid if $f(x)$ and $g(x)$ are continuous in $a \leqq x \leqq b$ and $g(x) \geqq 0$.

## LEIBNITZ'S RULE FOR DIFFERENTIATION OF INTEGRALS

**15.14** $$\frac{d}{d\alpha}\int_{\phi_1(\alpha)}^{\phi_2(\alpha)} F(x,\alpha)\,dx \;=\; \int_{\phi_1(\alpha)}^{\phi_2(\alpha)} \frac{\partial F}{\partial \alpha}dx \;+\; F(\phi_2,\alpha)\frac{d\phi_1}{d\alpha} \;-\; F(\phi_1,\alpha)\frac{d\phi_2}{d\alpha}$$

## APPROXIMATE FORMULAS FOR DEFINITE INTEGRALS

In the following the interval from $x = a$ to $x = b$ is subdivided into $n$ equal parts by the points $a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b$ and we let $y_0 = f(x_0)$, $y_1 = f(x_1)$, $y_2 = f(x_2)$, $\ldots$, $y_n = f(x_n)$, $h = (b-a)/n$.

**Rectangular formula**

**15.15** $$\int_a^b f(x)\,dx \;\approx\; h(y_0 + y_1 + y_2 + \cdots + y_{n-1})$$

**Trapezoidal formula**

**15.16** $$\int_a^b f(x)\,dx \;\approx\; \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)$$

**Simpson's formula (or parabolic formula) for $n$ even**

**15.17** $$\int_a^b f(x)\,dx \;\approx\; \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$

## DEFINITE INTEGRALS INVOLVING RATIONAL OR IRRATIONAL EXPRESSIONS

**15.18** $$\int_0^\infty \frac{dx}{x^2+a^2} \;=\; \frac{\pi}{2a}$$

**15.19** $$\int_0^\infty \frac{x^{p-1}\,dx}{1+x} \;=\; \frac{\pi}{\sin p\pi}, \qquad 0 < p < 1$$

**15.20** $$\int_0^\infty \frac{x^m\,dx}{x^n+a^n} \;=\; \frac{\pi a^{m+1-n}}{n\sin[(m+1)\pi/n]}, \qquad 0 < m+1 < n$$

**15.21** $$\int_0^\infty \frac{x^m\,dx}{1+2x\cos\beta+x^2} \;=\; \frac{\pi}{\sin m\pi}\,\frac{\sin m\beta}{\sin\beta}$$

**15.22** $$\int_0^a \frac{dx}{\sqrt{a^2-x^2}} \;=\; \frac{\pi}{2}$$

**15.23** $$\int_0^a \sqrt{a^2-x^2}\,dx \;=\; \frac{\pi a^2}{4}$$

**15.24** $$\int_0^a x^m(a^n-x^n)^p\,dx \;=\; \frac{a^{m+1+np}\,\Gamma[(m+1)/n]\,\Gamma(p+1)}{n\Gamma[(m+1)/n+p+1]}$$

**15.25** $$\int_0^\infty \frac{x^m\,dx}{(x^n+a^n)^r} \;=\; \frac{(-1)^{r-1}\pi a^{m+1-nr}\,\Gamma[(m+1)/n]}{n\sin[(m+1)\pi/n](r-1)!\,\Gamma[(m+1)/n-r+1]}, \qquad 0 < m+1 < nr$$

## DEFINITE INTEGRALS INVOLVING TRIGONOMETRIC FUNCTIONS

All letters are considered positive unless otherwise indicated.

**15.26** $$\int_0^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & m, n \text{ integers and } m \neq n \\ \pi/2 & m, n \text{ integers and } m = n \end{cases}$$

**15.27** $$\int_0^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & m, n \text{ integers and } m \neq n \\ \pi/2 & m, n \text{ integers and } m = n \end{cases}$$

**15.28** $$\int_0^{\pi} \sin mx \cos nx \, dx = \begin{cases} 0 & m, n \text{ integers and } m + n \text{ odd} \\ 2m/(m^2 - n^2) & m, n \text{ integers and } m + n \text{ even} \end{cases}$$

**15.29** $$\int_0^{\pi/2} \sin^2 x \, dx = \int_0^{\pi/2} \cos^2 x \, dx = \frac{\pi}{4}$$

**15.30** $$\int_0^{\pi/2} \sin^{2m} x \, dx = \int_0^{\pi/2} \cos^{2m} x \, dx = \frac{1 \cdot 3 \cdot 5 \cdots 2m-1}{2 \cdot 4 \cdot 6 \cdots 2m} \frac{\pi}{2}, \qquad m = 1, 2, \ldots$$

**15.31** $$\int_0^{\pi/2} \sin^{2m+1} x \, dx = \int_0^{\pi/2} \cos^{2m+1} x \, dx = \frac{2 \cdot 4 \cdot 6 \cdots 2m}{1 \cdot 3 \cdot 5 \cdots 2m+1}, \qquad m = 1, 2, \ldots$$

**15.32** $$\int_0^{\pi/2} \sin^{2p-1} x \cos^{2q-1} x \, dx = \frac{\Gamma(p)\,\Gamma(q)}{2\,\Gamma(p+q)}$$

**15.33** $$\int_0^{\infty} \frac{\sin px}{x} \, dx = \begin{cases} \pi/2 & p > 0 \\ 0 & p = 0 \\ -\pi/2 & p < 0 \end{cases}$$

**15.34** $$\int_0^{\infty} \frac{\sin px \cos qx}{x} \, dx = \begin{cases} 0 & p > q > 0 \\ \pi/2 & 0 < p < q \\ \pi/4 & p = q > 0 \end{cases}$$

**15.35** $$\int_0^{\infty} \frac{\sin px \sin qx}{x^2} \, dx = \begin{cases} \pi p/2 & 0 < p \leqq q \\ \pi q/2 & p \geqq q > 0 \end{cases}$$

**15.36** $$\int_0^{\infty} \frac{\sin^2 px}{x^2} \, dx = \frac{\pi p}{2}$$

**15.37** $$\int_0^{\infty} \frac{1 - \cos px}{x^2} \, dx = \frac{\pi p}{2}$$

**15.38** $$\int_0^{\infty} \frac{\cos px - \cos qx}{x} \, dx = \ln \frac{q}{p}$$

**15.39** $$\int_0^{\infty} \frac{\cos px - \cos qx}{x^2} \, dx = \frac{\pi(q-p)}{2}$$

**15.40** $$\int_0^{\infty} \frac{\cos mx}{x^2 + a^2} \, dx = \frac{\pi}{2a} e^{-ma}$$

**15.41** $$\int_0^{\infty} \frac{x \sin mx}{x^2 + a^2} \, dx = \frac{\pi}{2} e^{-ma}$$

**15.42** $$\int_0^{\infty} \frac{\sin mx}{x(x^2 + a^2)} \, dx = \frac{\pi}{2a^2}(1 - e^{-ma})$$

**15.43** $$\int_0^{2\pi} \frac{dx}{a + b \sin x} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

**15.44** $$\int_0^{2\pi} \frac{dx}{a + b \cos x} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

**15.45** $$\int_0^{\pi/2} \frac{dx}{a + b \cos x} = \frac{\cos^{-1}(b/a)}{\sqrt{a^2 - b^2}}$$

**15.46** $$\int_0^{2\pi} \frac{dx}{(a+b\sin x)^2} = \int_0^{2\pi} \frac{dx}{(a+b\cos x)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}$$

**15.47** $$\int_0^{2\pi} \frac{dx}{1-2a\cos x+a^2} = \frac{2\pi}{1-a^2}, \qquad 0<a<1$$

**15.48** $$\int_0^{\pi} \frac{x\sin x\,dx}{1-2a\cos x+a^2} = \begin{cases} (\pi/a)\ln(1+a) & |a|<1 \\ \pi\ln(1+1/a) & |a|>1 \end{cases}$$

**15.49** $$\int_0^{\pi} \frac{\cos mx\,dx}{1-2a\cos x+a^2} = \frac{\pi a^m}{1-a^2}, \quad a^2<1, \quad m=0,1,2,\ldots$$

**15.50** $$\int_0^{\infty} \sin ax^2\,dx = \int_0^{\infty} \cos ax^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}$$

**15.51** $$\int_0^{\infty} \sin ax^n\,dx = \frac{1}{na^{1/n}}\Gamma(1/n)\sin\frac{\pi}{2n}, \qquad n>1$$

**15.52** $$\int_0^{\infty} \cos ax^n\,dx = \frac{1}{na^{1/n}}\Gamma(1/n)\cos\frac{\pi}{2n}, \qquad n>1$$

**15.53** $$\int_0^{\infty} \frac{\sin x}{\sqrt{x}}\,dx = \int_0^{\infty} \frac{\cos x}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}$$

**15.54** $$\int_0^{\infty} \frac{\sin x}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\sin(p\pi/2)}, \qquad 0<p<1$$

**15.55** $$\int_0^{\infty} \frac{\cos x}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\cos(p\pi/2)}, \qquad 0<p<1$$

**15.56** $$\int_0^{\infty} \sin ax^2\cos 2bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left(\cos\frac{b^2}{a} - \sin\frac{b^2}{a}\right)$$

**15.57** $$\int_0^{\infty} \cos ax^2\cos 2bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left(\cos\frac{b^2}{a} + \sin\frac{b^2}{a}\right)$$

**15.58** $$\int_0^{\infty} \frac{\sin^3 x}{x^3}\,dx = \frac{3\pi}{8}$$

**15.59** $$\int_0^{\infty} \frac{\sin^4 x}{x^4}\,dx = \frac{\pi}{3}$$

**15.60** $$\int_0^{\infty} \frac{\tan x}{x}\,dx = \frac{\pi}{2}$$

**15.61** $$\int_0^{\pi/2} \frac{dx}{1+\tan^m x} = \frac{\pi}{4}$$

**15.62** $$\int_0^{\pi/2} \frac{x}{\sin x}\,dx = 2\left\{\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\cdots\right\}$$

**15.63** $$\int_0^{1} \frac{\tan^{-1} x}{x}\,dx = \frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\cdots$$ ✓ catalan

**15.64** $$\int_0^{1} \frac{\sin^{-1} x}{x}\,dx = \frac{\pi}{2}\ln 2$$

**15.65** $$\int_0^{1} \frac{1-\cos x}{x}\,dx - \int_1^{\infty} \frac{\cos x}{x}\,dx = \gamma$$

**15.66** $$\int_0^{\infty} \left(\frac{1}{1+x^2}-\cos x\right)\frac{dx}{x} = \gamma$$

**15.67** $$\int_0^{\infty} \frac{\tan^{-1} px - \tan^{-1} qx}{x}\,dx = \frac{\pi}{2}\ln\frac{p}{q}$$

## DEFINITE INTEGRALS INVOLVING EXPONENTIAL FUNCTIONS

**15.68** $$\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$$

**15.69** $$\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$$

**15.70** $$\int_0^\infty \frac{e^{-ax}\sin bx}{x}\,dx = \tan^{-1}\frac{b}{a}$$

**15.71** $$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx = \ln\frac{b}{a}$$

**15.72** $$\int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

**15.73** $$\int_0^\infty e^{-ax^2}\cos bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{-b^2/4a}$$

**15.74** $$\int_0^\infty e^{-(ax^2+bx+c)}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{(b^2-4ac)/4a}\,\operatorname{erfc}\frac{b}{2\sqrt{a}}$$

where $$\operatorname{erfc}(p) = \frac{2}{\sqrt{\pi}}\int_p^\infty e^{-x^2}\,dx$$

**15.75** $$\int_{-\infty}^\infty e^{-(ax^2+bx+c)}\,dx = \sqrt{\frac{\pi}{a}}\,e^{(b^2-4ac)/4a}$$

**15.76** $$\int_0^\infty x^n e^{-ax}\,dx = \frac{\Gamma(n+1)}{a^{n+1}}$$

**15.77** $$\int_0^\infty x^m e^{-ax^2}\,dx = \frac{\Gamma[(m+1)/2]}{2a^{(m+1)/2}}$$

**15.78** $$\int_0^\infty e^{-(ax^2+b/x^2)}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{-2\sqrt{ab}}$$

**15.79** $$\int_0^\infty \frac{x\,dx}{e^x-1} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots = \frac{\pi^2}{6}$$

**15.80** $$\int_0^\infty \frac{x^{n-1}}{e^x-1}\,dx = \Gamma(n)\left(\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\cdots\right)$$

For even $n$ this can be summed in terms of Bernoulli numbers [see pages 108-109 and 114-115].

**15.81** $$\int_0^\infty \frac{x\,dx}{e^x+1} = \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots = \frac{\pi^2}{12}$$

**15.82** $$\int_0^\infty \frac{x^{n-1}}{e^x+1}\,dx = \Gamma(n)\left(\frac{1}{1^n}-\frac{1}{2^n}+\frac{1}{3^n}-\cdots\right)$$

For some positive integer values of $n$ the series can be summed [see pages 108-109 and 114-115].

**15.83** $$\int_0^\infty \frac{\sin mx}{e^{2\pi x}-1}\,dx = \frac{1}{4}\coth\frac{m}{2}-\frac{1}{2m}$$

**15.84** $$\int_0^\infty \left(\frac{1}{1+x}-e^{-x}\right)\frac{dx}{x} = \gamma$$

**15.85** $$\int_0^\infty \frac{e^{-x^2}-e^{-x}}{x}\,dx = \tfrac{1}{2}\gamma$$

**15.86** $$\int_0^\infty \left(\frac{1}{e^x-1}-\frac{e^{-x}}{x}\right)dx = \gamma$$

**15.87** $$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x\sec px}\,dx = \frac{1}{2}\ln\left(\frac{b^2+p^2}{a^2+p^2}\right)$$

**15.88** $$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x\csc px}\,dx = \tan^{-1}\frac{b}{p} - \tan^{-1}\frac{a}{p}$$

**15.89** $$\int_0^\infty \frac{e^{-ax}(1-\cos x)}{x^2}\,dx = \cot^{-1}a - \frac{a}{2}\ln(a^2+1)$$

## DEFINITE INTEGRALS INVOLVING LOGARITHMIC FUNCTIONS

**15.90** $$\int_0^1 x^m(\ln x)^n\,dx = \frac{(-1)^n n!}{(m+1)^{n+1}} \qquad m>-1,\ n=0,1,2,\ldots$$

If $n \neq 0,1,2,\ldots$ replace $n!$ by $\Gamma(n+1)$.

**15.91** $$\int_0^1 \frac{\ln x}{1+x}\,dx = -\frac{\pi^2}{12}$$

**15.92** $$\int_0^1 \frac{\ln x}{1-x}\,dx = -\frac{\pi^2}{6}$$

**15.93** $$\int_0^1 \frac{\ln(1+x)}{x}\,dx = \frac{\pi^2}{12}$$

**15.94** $$\int_0^1 \frac{\ln(1-x)}{x}\,dx = -\frac{\pi^2}{6}$$

**15.95** $$\int_0^1 \ln x\ln(1+x)\,dx = 2 - 2\ln 2 - \frac{\pi^2}{12}$$

**15.96** $$\int_0^1 \ln x\ln(1-x)\,dx = 2 - \frac{\pi^2}{6}$$

**15.97** $$\int_0^\infty \frac{x^{p-1}\ln x}{1+x}\,dx = -\pi^2\csc p\pi\cot p\pi \qquad 0<p<1$$

**15.98** $$\int_0^1 \frac{x^m-x^n}{\ln x}\,dx = \ln\frac{m+1}{n+1}$$

→**15.99** $$\int_0^\infty e^{-x}\ln x\,dx = -\gamma$$

**15.100** $$\int_0^\infty e^{-x^2}\ln x\,dx = -\frac{\sqrt{\pi}}{4}(\gamma+2\ln 2)$$

**15.101** $$\int_0^\infty \ln\left(\frac{e^x+1}{e^x-1}\right)dx = \frac{\pi^2}{4}$$

**15.102** $$\int_0^{\pi/2} \ln\sin x\,dx = \int_0^{\pi/2}\ln\cos x\,dx = -\frac{\pi}{2}\ln 2$$

**15.103** $$\int_0^{\pi/2} (\ln\sin x)^2\,dx = \int_0^{\pi/2}(\ln\cos x)^2\,dx = \frac{\pi}{2}(\ln 2)^2 + \frac{\pi^3}{24}$$

**15.104** $$\int_0^{\pi} x\ln\sin x\,dx = -\frac{\pi^2}{2}\ln 2$$

**15.105** $$\int_0^{\pi/2} \sin x\ln\sin x\,dx = \ln 2 - 1$$

**15.106** $$\int_0^{2\pi} \ln(a+b\sin x)\,dx = \int_0^{2\pi}\ln(a+b\cos x)\,dx = 2\pi\ln(a+\sqrt{a^2-b^2})$$

**15.107** $$\int_0^{\pi} \ln(a + b\cos x)\,dx = \pi \ln\left(\frac{a+\sqrt{a^2-b^2}}{2}\right)$$

**15.108** $$\int_0^{\pi} \ln(a^2 - 2ab\cos x + b^2)\,dx = \begin{cases} 2\pi \ln a, & a \geqq b > 0 \\ 2\pi \ln b, & b \geqq a > 0 \end{cases}$$

**15.109** $$\int_0^{\pi/4} \ln(1 + \tan x)\,dx = \frac{\pi}{8}\ln 2$$

**15.110** $$\int_0^{\pi/2} \sec x \ln\left(\frac{1 + b\cos x}{1 + a\cos x}\right) dx = \tfrac{1}{2}\{(\cos^{-1} a)^2 - (\cos^{-1} b)^2\}$$

**15.111** $$\int_0^{a} \ln\left(2\sin\frac{x}{2}\right) dx = -\left(\frac{\sin a}{1^2} + \frac{\sin 2a}{2^2} + \frac{\sin 3a}{3^2} + \cdots\right)$$

See also 15.102.

## DEFINITE INTEGRALS INVOLVING HYPERBOLIC FUNCTIONS

**15.112** $$\int_0^{\infty} \frac{\sin ax}{\sinh bx}\,dx = \frac{\pi}{2b}\tanh\frac{a\pi}{2b}$$

**15.113** $$\int_0^{\infty} \frac{\cos ax}{\cosh bx}\,dx = \frac{\pi}{2b}\operatorname{sech}\frac{a\pi}{2b}$$

**15.114** $$\int_0^{\infty} \frac{x\,dx}{\sinh ax} = \frac{\pi^2}{4a^2}$$

**15.115** $$\int_0^{\infty} \frac{x^n\,dx}{\sinh ax} = \frac{2^{n+1}-1}{2^n a^{n+1}}\Gamma(n+1)\left\{\frac{1}{1^{n+1}} + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \cdots\right\}$$

If $n$ is an odd positive integer, the series can be summed [see page 108].

**15.116** $$\int_0^{\infty} \frac{\sinh ax}{e^{bx}+1}\,dx = \frac{\pi}{2b}\csc\frac{a\pi}{b} - \frac{1}{2a}$$

**15.117** $$\int_0^{\infty} \frac{\sinh ax}{e^{bx}-1}\,dx = \frac{1}{2a} - \frac{\pi}{2b}\cot\frac{a\pi}{b}$$

## MISCELLANEOUS DEFINITE INTEGRALS

**15.118** $$\int_0^{\infty} \frac{f(ax)-f(bx)}{x}\,dx = \{f(0) - f(\infty)\}\ln\frac{b}{a}$$

This is called *Frullani's integral.* It holds if $f'(x)$ is continuous and $\int_1^{\infty} \frac{f(x)-f(\infty)}{x}\,dx$ converges.

**15.119** $$\int_0^{1} \frac{dx}{x^x} = \frac{1}{1^1} + \frac{1}{2^2} + \frac{1}{3^3} + \cdots$$

**15.120** $$\int_{-a}^{a} (a+x)^{m-1}(a-x)^{n-1}\,dx = (2a)^{m+n-1}\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

# 16 THE GAMMA FUNCTION

## DEFINITION OF THE GAMMA FUNCTION $\Gamma(n)$ FOR $n > 0$

**16.1** $$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}\,dt \qquad n > 0$$

## RECURSION FORMULA

**16.2** $$\Gamma(n+1) = n\,\Gamma(n)$$

**16.3** $$\Gamma(n+1) = n! \qquad \text{if } n = 0, 1, 2, \ldots \text{ where } 0! = 1$$

## THE GAMMA FUNCTION FOR $n < 0$

For $n < 0$ the gamma function can be defined by using 16.2, i.e.

**16.4** $$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

## GRAPH OF THE GAMMA FUNCTION

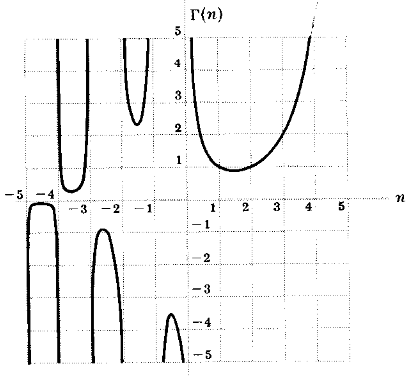


Fig. 16-1

## SPECIAL VALUES FOR THE GAMMA FUNCTION

**16.5** $$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

**16.6** $$\Gamma(m+\tfrac{1}{2}) = \frac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{2^m}\sqrt{\pi} \qquad m = 1, 2, 3, \ldots = \frac{(2m)!\,\sqrt{\pi}}{4^m m!}$$

**16.7** $$\Gamma(-m+\tfrac{1}{2}) = \frac{(-1)^m 2^m \sqrt{\pi}}{1 \cdot 3 \cdot 5 \cdots (2m-1)} \qquad m = 1, 2, 3, \ldots$$

## RELATIONSHIPS AMONG GAMMA FUNCTIONS

**16.8**
$$\Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin p\pi}$$

**16.9**
$$2^{2x-1}\,\Gamma(x)\,\Gamma(x+\tfrac{1}{2}) = \sqrt{\pi}\,\Gamma(2x)$$

This is called the *duplication formula.*

**16.10**
$$\Gamma(x)\,\Gamma\left(x+\frac{1}{m}\right)\Gamma\left(x+\frac{2}{m}\right)\cdots\Gamma\left(x+\frac{m-1}{m}\right) = m^{\frac{1}{2}-mx}\,(2\pi)^{(m-1)/2}\,\Gamma(mx)$$

For $m = 2$ this reduces to 16.9.

## OTHER DEFINITIONS OF THE GAMMA FUNCTION

**16.11**
$$\Gamma(x+1) = \lim_{k\to\infty} \frac{1\cdot 2\cdot 3\cdots k}{(x+1)(x+2)\cdots(x+k)}\,k^x$$

**16.12**
$$\frac{1}{\Gamma(x)} = xe^{\gamma x}\prod_{m=1}^{\infty}\left\{\left(1+\frac{x}{m}\right)e^{-x/m}\right\}$$

This is an infinite product representation for the gamma function where $\gamma$ is Euler's constant.

## DERIVATIVES OF THE GAMMA FUNCTION

**16.13**
$$\Gamma'(1) = \int_0^\infty e^{-x}\ln x\,dx = -\gamma$$

**16.14**
$$\frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \left(\frac{1}{1}-\frac{1}{x}\right) + \left(\frac{1}{2}-\frac{1}{x+1}\right) + \cdots + \left(\frac{1}{n}-\frac{1}{x+n-1}\right) + \cdots$$

## ASYMPTOTIC EXPANSIONS FOR THE GAMMA FUNCTION

**16.15**
$$\Gamma(x+1) = \sqrt{2\pi x}\,x^x e^{-x}\left\{1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51{,}840x^3} + \cdots\right\}$$

This is called *Stirling's asymptotic series.*

If we let $x = n$ a positive integer in 16.15, then a useful approximation for $n!$ where $n$ is large [e.g. $n > 10$] is given by *Stirling's formula*

**16.16**
$$n! \sim \sqrt{2\pi n}\,n^n e^{-n}$$

where $\sim$ is used to indicate that the ratio of the terms on each side approaches 1 as $n \to \infty$.

## MISCELLANEOUS RESULTS

**16.17**
$$|\Gamma(ix)|^2 = \frac{\pi}{x\sinh \pi x}$$

# 17 THE BETA FUNCTION

## DEFINITION OF THE BETA FUNCTION $B(m,n)$

**17.1** $$B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,dt \qquad m>0,\ n>0$$

## RELATIONSHIP OF BETA FUNCTION TO GAMMA FUNCTION

**17.2** $$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Extensions of $B(m,n)$ to $m<0,\ n<0$ is provided by using 16.4, page 101.

## SOME IMPORTANT RESULTS

**17.3** $$B(m,n) = B(n,m)$$

**17.4** $$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$$

**17.5** $$B(m,n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\,dt$$

**17.6** $$B(m,n) = r^n(r+1)^m\int_0^1 \frac{t^{m-1}(1-t)^{n-1}}{(r+t)^{m+n}}\,dt$$

# 18

# BASIC DIFFERENTIAL EQUATIONS and SOLUTIONS

| DIFFERENTIAL EQUATION | SOLUTION |
|---|---|
| **18.1** Separation of variables | $\int \frac{f_1(x)}{f_2(x)}\,dx + \int \frac{g_2(y)}{g_1(y)}\,dy = c$ |
| $f_1(x)\,g_1(y)\,dx + f_2(x)\,g_2(y)\,dy = 0$ | |
| **18.2** Linear first order equation | $y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$ |
| $\frac{dy}{dx} + P(x)y = Q(x)$ | |
| **18.3** Bernoulli's equation | $v\,e^{(1-n)\int P\,dx} = (1-n)\int Q\,e^{(1-n)\int P\,dx}\,dx + c$<br>where $v = y^{1-n}$. If $n = 1$, the solution is<br>$\ln y = \int (Q-P)\,dx + c$ |
| $\frac{dy}{dx} + P(x)y = Q(x)y^n$ | |
| **18.4** Exact equation | $\int M\,\partial x + \int \left(N - \frac{\partial}{\partial y}\int M\,\partial x\right)dy = c$<br>where $\partial x$ indicates that the integration is to be performed with respect to $x$ keeping $y$ constant. |
| $M(x,y)\,dx + N(x,y)\,dy = 0$<br>where $\partial M/\partial y = \partial N/\partial x$. | |
| **18.5** Homogeneous equation | $\ln x = \int \frac{dv}{F(v) - v} + c$<br>where $v = y/x$. If $F(v) = v$, the solution is $y = cx$. |
| $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ | |

| **DIFFERENTIAL EQUATION** | **SOLUTION** |
|---|---|
| **18.6**<br>$yF(xy)\,dx + xG(xy)\,dy = 0$ | $\ln x = \int \frac{G(v)\,dv}{v\{G(v)-F(v)\}} + c$<br>where $v = xy$. If $G(v) = F(v)$, the solution is $xy = c$. |
| **18.7** Linear, homogeneous second order equation<br>$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$<br>$a, b$ are real constants. | Let $m_1, m_2$ be the roots of $m^2 + am + b = 0$. Then there are 3 cases.<br>Case 1. $m_1, m_2$ real and distinct:<br>$y = c_1e^{m_1x} + c_2e^{m_2x}$<br>Case 2. $m_1, m_2$ real and equal:<br>$y = c_1e^{m_1x} + c_2xe^{m_1x}$<br>Case 3. $m_1 = p + qi$, $m_2 = p - qi$:<br>$y = e^{px}(c_1\cos qx + c_2\sin qx)$<br>where $p = -a/2$, $q = \sqrt{b - a^2/4}$. |
| **18.8** Linear, nonhomogeneous second order equation<br>$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = R(x)$<br>$a, b$ are real constants. | There are 3 cases corresponding to those of entry 18.7 above.<br>Case 1.<br>$y = c_1e^{m_1x} + c_2e^{m_2x} + \frac{e^{m_1x}}{m_1 - m_2}\int e^{-m_1x}R(x)\,dx + \frac{e^{m_2x}}{m_2 - m_1}\int e^{-m_2x}R(x)\,dx$<br>Case 2.<br>$y = c_1e^{m_1x} + c_2xe^{m_1x} + xe^{m_1x}\int e^{-m_1x}R(x)\,dx - e^{m_1x}\int xe^{-m_1x}R(x)\,dx$<br>Case 3.<br>$y = e^{px}(c_1\cos qx + c_2\sin qx) + \frac{e^{px}\sin qx}{q}\int e^{-px}R(x)\cos qx\,dx - \frac{e^{px}\cos qx}{q}\int e^{-px}R(x)\sin qx\,dx$ |
| **18.9** Euler or Cauchy equation<br>$x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = S(x)$ | Putting $x = e^t$, the equation becomes<br>$\frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = S(e^t)$<br>and can then be solved as in entries 18.7 and 18.8 above. |

| DIFFERENTIAL EQUATION | SOLUTION |
|---|---|
| **18.10** Bessel's equation | $y = c_1 J_n(\lambda x) + c_2 Y_n(x)$<br>See pages 136-137. |
| $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (\lambda^2 x^2 - n^2)y = 0$ | |
| **18.11** Transformed Bessel's equation | $y = x^{-p}\left\{c_1 J_{q/r}\left(\dfrac{\alpha}{r}x^r\right) + c_2 Y_{q/r}\left(\dfrac{\alpha}{r}x^r\right)\right\}$<br>where $q = \sqrt{p^2 - \beta^2}$. |
| $x^2 \dfrac{d^2y}{dx^2} + (2p+1)x\dfrac{dy}{dx} + (\alpha^2 x^{2r} + \beta^2)y = 0$ | |
| **18.12** Legendre's equation | $y = c_1 P_n(x) + c_2 Q_n(x)$<br>See pages 146-148. |
| $(1 - x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + n(n+1)y = 0$ | |

# 19

# SERIES of CONSTANTS

## ARITHMETIC SERIES

**19.1** $$a + (a+d) + (a+2d) + \cdots + \{a + (n-1)d\} = \tfrac{1}{2}n\{2a + (n-1)d\} = \tfrac{1}{2}n(a+l)$$

where $l = a + (n-1)d$ is the last term.

Some special cases are

**19.2** $$1 + 2 + 3 + \cdots + n = \tfrac{1}{2}n(n+1)$$

**19.3** $$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

## GEOMETRIC SERIES

**19.4** $$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r} = \frac{a-rl}{1-r}$$

where $l = ar^{n-1}$ is the last term and $r \neq 1$.

If $-1 < r < 1$, then

**19.5** $$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}$$

## ARITHMETIC-GEOMETRIC SERIES

**19.6** $$a + (a+d)r + (a+2d)r^2 + \cdots + \{a + (n-1)d\}r^{n-1} = \frac{a(1-r^n)}{1-r} + \frac{rd\{1 - nr^{n-1} + (n-1)r^n\}}{(1-r)^2}$$

where $r \neq 1$.

If $-1 < r < 1$, then

**19.7** $$a + (a+d)r + (a+2d)r^2 + \cdots = \frac{a}{1-r} + \frac{rd}{(1-r)^2}$$

## SUMS OF POWERS OF POSITIVE INTEGERS

**19.8** $$1^p + 2^p + 3^p + \cdots + n^p = \frac{n^{p+1}}{p+1} + \tfrac{1}{2}n^p + \frac{B_1pn^{p-1}}{2!} - \frac{B_2p(p-1)(p-2)n^{p-3}}{4!} + \cdots$$

where the series terminates at $n^2$ or $n$ according as $p$ is odd or even, and $B_k$ are the *Bernoulli numbers* [see page 114].

Some special cases are

**19.9** $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$

**19.10** $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

**19.11** $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4} = (1+2+3+\cdots+n)^2$

**19.12** $1^4 + 2^4 + 3^4 + \cdots + n^4 = \dfrac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

If $S_k = 1^k + 2^k + 3^k + \cdots + n^k$ where $k$ and $n$ are positive integers, then

**19.13** $\dbinom{k+1}{1}S_1 + \dbinom{k+1}{2}S_2 + \cdots + \dbinom{k+1}{k}S_k = (n+1)^{k+1} - (n+1)$

## SERIES INVOLVING RECIPROCALS OF POWERS OF POSITIVE INTEGERS

**19.14** $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln 2$ 0.6931471806

**19.15** $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \frac{\pi}{4}$ 0.7853981635

**19.16** $1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \frac{1}{13} - \cdots = \frac{\pi\sqrt{3}}{9} + \frac{1}{3}\ln 2$ 0.8356488485

**19.17** $1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \frac{1}{17} - \cdots = \frac{\pi\sqrt{2}}{8} + \frac{\sqrt{2}\ln(1+\sqrt{2})}{4}$ 0.8669729873

**19.18** $\frac{1}{2} - \frac{1}{5} + \frac{1}{8} - \frac{1}{11} + \frac{1}{14} - \cdots = \frac{\pi\sqrt{3}}{9} - \frac{1}{3}\ln 2$ 0.3735507281

**19.19** $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$ 1.644934067

**19.20** $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}$ 1.082323234

**19.21** $\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \cdots = \frac{\pi^6}{945}$ 1.017343063

**19.22** $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$ 0.8224670337

**19.23** $\frac{1}{1^4} - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \cdots = \frac{7\pi^4}{720}$ 0.9470328300

**19.24** $\frac{1}{1^6} - \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} + \cdots = \frac{31\pi^6}{30{,}240}$ 0.9855510919

**19.25** $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$ 1.233700551

**19.26** $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96}$ 1.014678032

**19.27** $\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \cdots = \frac{\pi^6}{960}$ 1.001447077

**19.28** $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}$ 0.9689461466

**19.29** $\frac{1}{1^3} + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{3\pi^3\sqrt{2}}{128}$ 1.027722586

**19.30** $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \frac{1}{7\cdot 9} + \cdots = \frac{1}{2}$

**19.31** $\frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} + \frac{1}{4\cdot 6} + \cdots = \frac{3}{4}$

**19.32** $\frac{1}{1^2\cdot 3^2} + \frac{1}{3^2\cdot 5^2} + \frac{1}{5^2\cdot 7^2} + \frac{1}{7^2\cdot 9^2} + \cdots = \frac{\pi^2-8}{16}$ 0.1168502753

$$\int_0^1 \frac{u^{a-1}}{1-u^d}\,du = \frac{1}{a}+\frac{1}{a+b}+\frac{1}{a+2b}+\frac{1}{a+3b}+\frac{1}{a+4b}+\cdots$$ diverge

**19.33** $$\frac{1}{1^2\cdot 2^2\cdot 3^2}+\frac{1}{2^2\cdot 3^2\cdot 4^2}+\frac{1}{3^2\cdot 4^2\cdot 5^2}+\cdots = \frac{4\pi^2-39}{16}$$

**19.34** $$\frac{1}{a}-\frac{1}{a+d}+\frac{1}{a+2d}-\frac{1}{a+3d}+\cdots = \int_0^1 \frac{u^{a-1}\,du}{1+u^d}$$ * Ok

**19.35** $$\frac{1}{1^{2p}}+\frac{1}{2^{2p}}+\frac{1}{3^{2p}}+\frac{1}{4^{2p}}+\cdots = \frac{2^{2p-1}\pi^{2p}B_p}{(2p)!}$$

**19.36** $$\frac{1}{1^{2p}}+\frac{1}{3^{2p}}+\frac{1}{5^{2p}}+\frac{1}{7^{2p}}+\cdots = \frac{(2^{2p}-1)\pi^{2p}B_p}{2(2p)!}$$

**19.37** $$\frac{1}{1^{2p}}-\frac{1}{2^{2p}}+\frac{1}{3^{2p}}-\frac{1}{4^{2p}}+\cdots = \frac{(2^{2p-1}-1)\pi^{2p}B_p}{(2p)!}$$

**19.38** $$\frac{1}{1^{2p+1}}-\frac{1}{3^{2p+1}}+\frac{1}{5^{2p+1}}-\frac{1}{7^{2p+1}}+\cdots = \frac{\pi^{2p+1}E_p}{2^{2p+2}(2p)!}$$

## MISCELLANEOUS SERIES

**19.39** $$\frac{1}{2}+\cos\alpha+\cos 2\alpha+\cdots+\cos n\alpha = \frac{\sin(n+\frac{1}{2})\alpha}{2\sin(\alpha/2)}$$

**19.40** $$\sin\alpha+\sin 2\alpha+\sin 3\alpha+\cdots+\sin n\alpha = \frac{\sin[\frac{1}{2}(n+1)]\alpha\,\sin\frac{1}{2}n\alpha}{\sin(\alpha/2)}$$

**19.41** $$1+r\cos\alpha+r^2\cos 2\alpha+r^3\cos 3\alpha+\cdots = \frac{1-r\cos\alpha}{1-2r\cos\alpha+r^2},\quad |r|<1$$

**19.42** $$r\sin\alpha+r^2\sin 2\alpha+r^3\sin 3\alpha+\cdots = \frac{r\sin\alpha}{1-2r\cos\alpha+r^2},\quad |r|<1$$

**19.43** $$1+r\cos\alpha+r^2\cos 2\alpha+\cdots+r^n\cos n\alpha = \frac{r^{n+2}\cos n\alpha-r^{n+1}\cos(n+1)\alpha-r\cos\alpha+1}{1-2r\cos\alpha+r^2}$$

**19.44** $$r\sin\alpha+r^2\sin 2\alpha+\cdots+r^n\sin n\alpha = \frac{r\sin\alpha-r^{n+1}\sin(n+1)\alpha+r^{n+2}\sin n\alpha}{1-2r\cos\alpha+r^2}$$

## THE EULER-MACLAURIN SUMMATION FORMULA

*

**19.45** $$\begin{aligned}\sum_{k=1}^{n-1}F(k) &= \int_0^n F(k)\,dk-\frac{1}{2}\{F(0)+F(n)\}\\ &+\frac{1}{12}\{F'(n)-F'(0)\}-\frac{1}{720}\{F'''(n)-F'''(0)\}\\ &+\frac{1}{30{,}240}\{F^{(\mathrm{v})}(n)-F^{(\mathrm{v})}(0)\}-\frac{1}{1{,}209{,}600}\{F^{(\mathrm{vii})}(n)-F^{(\mathrm{vii})}(0)\}\\ &+\cdots(-1)^{p-1}\frac{B_p}{(2p)!}\{F^{(2p-1)}(n)-F^{(2p-1)}(0)\}+\cdots\end{aligned}$$

## THE POISSON SUMMATION FORMULA

**19.46** $$\sum_{k=-\infty}^{\infty}F(k) = \sum_{m=-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}e^{2\pi imx}F(x)\,dx\right\}$$

# 20 TAYLOR SERIES

## TAYLOR SERIES FOR FUNCTIONS OF ONE VARIABLE

**20.1** $$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^{(n-1)}(a)(x-a)^{n-1}}{(n-1)!} + R_n$$

where $R_n$, the remainder after $n$ terms, is given by either of the following forms:

**20.2** Lagrange's form $$R_n = \frac{f^{(n)}(\xi)(x-a)^n}{n!}$$

**20.3** Cauchy's form $$R_n = \frac{f^{(n)}(\xi)(x-\xi)^{n-1}(x-a)}{(n-1)!}$$

The value $\xi$, which may be different in the two forms, lies between $a$ and $x$. The result holds if $f(x)$ has continuous derivatives of order $n$ at least.

If $\lim_{n\to\infty} R_n = 0$, the infinite series obtained is called the *Taylor series* for $f(x)$ about $x = a$. If $a = 0$ the series is often called a *Maclaurin series*. These series, often called power series, generally converge for all values of $x$ in some interval called the *interval of convergence* and diverge for all $x$ outside this interval.

## BINOMIAL SERIES

**20.4** $$\begin{aligned}(a+x)^n &= a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \cdots \\ &= a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \binom{n}{3}a^{n-3}x^3 + \cdots\end{aligned}$$

Special cases are

**20.5** $$(a+x)^2 = a^2 + 2ax + x^2$$

**20.6** $$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3$$

**20.7** $$(a+x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$$

**20.8** $$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots \qquad -1 < x < 1$$

**20.9** $$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots \qquad -1 < x < 1$$

**20.10** $$(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + 15x^4 - \cdots \qquad -1 < x < 1$$

**20.11** $$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \cdots \qquad -1 < x \leqq 1$$

**20.12** $$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{2\cdot 4}x^2 + \frac{1\cdot 3}{2\cdot 4\cdot 6}x^3 - \cdots \qquad -1 < x \leqq 1$$

**20.13** $$(1+x)^{-1/3} = 1 - \frac{1}{3}x + \frac{1\cdot 4}{3\cdot 6}x^2 - \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}x^3 + \cdots \qquad -1 < x \leqq 1$$

**20.14** $$(1+x)^{1/3} = 1 + \frac{1}{3}x - \frac{2}{3\cdot 6}x^2 + \frac{2\cdot 5}{3\cdot 6\cdot 9}x^3 - \cdots \qquad -1 < x \leqq 1$$

## SERIES FOR EXPONENTIAL AND LOGARITHMIC FUNCTIONS

**20.15** $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad -\infty < x < \infty$$

**20.16** $$a^x = e^{x \ln a} = 1 + x \ln a + \frac{(x \ln a)^2}{2!} + \frac{(x \ln a)^3}{3!} + \cdots \qquad -\infty < x < \infty$$

**20.17** $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad -1 < x \leqq 1$$

**20.18** $$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots \qquad -1 < x < 1$$

**20.19** $$\ln x = 2\left\{\left(\frac{x-1}{x+1}\right) + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots\right\} \qquad x > 0$$

**20.20** $$\ln x = \left(\frac{x-1}{x}\right) + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \cdots \qquad x \geqq \tfrac{1}{2}$$

## SERIES FOR TRIGONOMETRIC FUNCTIONS

**20.21** $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad -\infty < x < \infty$$

**20.22** $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad -\infty < x < \infty$$

**20.23** $$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots + \frac{2^{2n}(2^{2n}-1)B_n x^{2n-1}}{(2n)!} + \cdots \qquad |x| < \frac{\pi}{2}$$

**20.24** $$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \cdots - \frac{2^{2n}B_n x^{2n-1}}{(2n)!} - \cdots \qquad 0 < |x| < \pi$$

**20.25** $$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots + \frac{E_n x^{2n}}{(2n)!} + \cdots \qquad |x| < \frac{\pi}{2}$$

**20.26** $$\csc x = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15{,}120} + \cdots + \frac{2(2^{2n-1}-1)B_n x^{2n-1}}{(2n)!} + \cdots \qquad 0 < |x| < \pi$$

**20.27** $$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \cdots \qquad |x| < 1$$

**20.28** $$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x = \frac{\pi}{2} - \left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \cdots\right) \qquad |x| < 1$$

**20.29** $$\tan^{-1} x = \begin{cases} x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots & |x| < 1 \\ \pm\dfrac{\pi}{2} - \dfrac{1}{x} + \dfrac{1}{3x^3} - \dfrac{1}{5x^5} + \cdots & [+ \text{ if } x \geqq 1,\ - \text{ if } x \leqq -1] \end{cases}$$

**20.30** $$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x = \begin{cases} \dfrac{\pi}{2} - \left(x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \cdots\right) & |x| < 1 \\ p\pi + \dfrac{1}{x} - \dfrac{1}{3x^3} + \dfrac{1}{5x^5} - \cdots & [p = 0 \text{ if } x > 1,\ p = 1 \text{ if } x < -1] \end{cases}$$

**20.31** $$\sec^{-1} x = \cos^{-1}(1/x) = \frac{\pi}{2} - \left(\frac{1}{x} + \frac{1}{2 \cdot 3x^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5x^5} + \cdots\right) \qquad |x| > 1$$

**20.32** $$\csc^{-1} x = \sin^{-1}(1/x) = \frac{1}{x} + \frac{1}{2 \cdot 3x^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5x^5} + \cdots \qquad |x| > 1$$

## SERIES FOR HYPERBOLIC FUNCTIONS

**20.33** $$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \qquad -\infty < x < \infty$$

**20.34** $$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \qquad -\infty < x < \infty$$

**20.35** $$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_n x^{2n-1}}{(2n)!} + \cdots \qquad |x| < \frac{\pi}{2}$$

**20.36** $$\coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} + \cdots \frac{(-1)^{n-1}2^{2n}B_n x^{2n-1}}{(2n)!} + \cdots \qquad 0 < |x| < \pi$$

**20.37** $$\operatorname{sech} x = 1 - \frac{x^2}{2} + \frac{5x^4}{24} - \frac{61x^6}{720} + \cdots \frac{(-1)^n E_n x^{2n}}{(2n)!} + \cdots \qquad |x| < \frac{\pi}{2}$$

**20.38** $$\operatorname{csch} x = \frac{1}{x} - \frac{x}{6} + \frac{7x^3}{360} - \frac{31x^5}{15{,}120} + \cdots \frac{(-1)^n 2(2^{2n-1}-1)B_n x^{2n-1}}{(2n)!} + \cdots \qquad 0 < |x| < \pi$$

**20.39** $$\sinh^{-1} x = \begin{cases} x - \dfrac{x^3}{2\cdot 3} + \dfrac{1\cdot 3x^5}{2\cdot 4\cdot 5} - \dfrac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7} + \cdots & |x| < 1 \\ \pm\left(\ln|2x| + \dfrac{1}{2\cdot 2x^2} - \dfrac{1\cdot 3}{2\cdot 4\cdot 4x^4} + \dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6x^6} - \cdots\right) & \begin{bmatrix} + \text{ if } x \geqq 1 \\ - \text{ if } x \leqq -1 \end{bmatrix} \end{cases}$$

**20.40** $$\cosh^{-1} x = \pm\left\{\ln(2x) - \left(\frac{1}{2\cdot 2x^2} + \frac{1\cdot 3}{2\cdot 4\cdot 4x^4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6x^6} + \cdots\right)\right\} \qquad \begin{bmatrix} + \text{ if } \cosh^{-1} x > 0,\ x \geqq 1 \\ - \text{ if } \cosh^{-1} x < 0,\ x \geqq 1 \end{bmatrix}$$

**20.41** $$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots \qquad |x| < 1$$

**20.42** $$\coth^{-1} x = \frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \frac{1}{7x^7} + \cdots \qquad |x| > 1$$

## MISCELLANEOUS SERIES

**20.43** $$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^5}{15} + \cdots \qquad -\infty < x < \infty$$

**20.44** $$e^{\cos x} = e\left(1 - \frac{x^2}{2} + \frac{x^4}{6} - \frac{31x^6}{720} + \cdots\right) \qquad -\infty < x < \infty$$

**20.45** $$e^{\tan x} = 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{3x^4}{8} + \cdots \qquad |x| < \frac{\pi}{2}$$

**20.46** $$e^x \sin x = x + x^2 + \frac{2x^3}{3} - \frac{x^5}{30} - \frac{x^6}{90} + \cdots + \frac{2^{n/2}\sin(n\pi/4)\,x^n}{n!} + \cdots \qquad -\infty < x < \infty$$

**20.47** $$e^x \cos x = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots + \frac{2^{n/2}\cos(n\pi/4)\,x^n}{n!} + \cdots \qquad -\infty < x < \infty$$

**20.48** $$\ln|\sin x| = \ln|x| - \frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} - \cdots - \frac{2^{2n-1}B_n x^{2n}}{n(2n)!} + \cdots \qquad 0 < |x| < \pi$$

**20.49** $$\ln|\cos x| = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17x^8}{2520} - \cdots - \frac{2^{2n-1}(2^{2n}-1)B_n x^{2n}}{n(2n)!} + \cdots \qquad |x| < \frac{\pi}{2}$$

**20.50** $$\ln|\tan x| = \ln|x| + \frac{x^2}{3} + \frac{7x^4}{90} + \frac{62x^6}{2835} + \cdots + \frac{2^{2n}(2^{2n-1}-1)B_n x^{2n}}{n(2n)!} + \cdots \qquad 0 < |x| < \frac{\pi}{2}$$

**20.51** $$\frac{\ln(1+x)}{1+x} = x - (1+\tfrac{1}{2})x^2 + (1+\tfrac{1}{2}+\tfrac{1}{3})x^3 - \cdots \qquad |x| < 1$$

## REVERSION OF POWER SERIES

If

**20.52** $$y = c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + \cdots$$

then

**20.53** $$x = C_1y + C_2y^2 + C_3y^3 + C_4y^4 + C_5y^5 + C_6y^6 + \cdots$$

where

**20.54** $$c_1C_1 = 1$$

**20.55** $$c_1^3C_2 = -c_2$$

**20.56** $$c_1^5C_3 = 2c_2^2 - c_1c_3$$

**20.57** $$c_1^7C_4 = 5c_1c_2c_3 - 5c_2^3 - c_1^2c_4$$

**20.58** $$c_1^9C_5 = 6c_1^2c_2c_4 + 3c_1^2c_3^2 - c_1^3c_5 + 14c_2^4 - 21c_1c_2^2c_3$$

**20.59** $$c_1^{11}C_6 = 7c_1^3c_2c_5 + 84c_1c_2^3c_3 + 7c_1^3c_3c_4 - 28c_1^2c_2c_3^2 - c_1^4c_6 - 28c_1^2c_2^2c_4 - 42c_2^5$$

## TAYLOR SERIES FOR FUNCTIONS OF TWO VARIABLES

**20.60** $$\begin{aligned} f(x,y) &= f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) \\ &\quad + \frac{1}{2!}\{(x-a)^2f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2f_{yy}(a,b)\} + \cdots \end{aligned}$$

where $f_x(a,b), f_y(a,b), \ldots$ denote partial derivatives with respect to $x, y, \ldots$ evaluated at $x = a$, $y = b$.

# 21

# BERNOULLI and EULER NUMBERS

## DEFINITION OF BERNOULLI NUMBERS

The *Bernoulli numbers* $B_1, B_2, B_3, \ldots$ are defined by the series

**21.1** $$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{B_1 x^2}{2!} - \frac{B_2 x^4}{4!} + \frac{B_3 x^6}{6!} - \cdots \qquad |x| < 2\pi$$

**21.2** $$1 - \frac{x}{2} \cot \frac{x}{2} = \frac{B_1 x^2}{2!} + \frac{B_2 x^4}{4!} + \frac{B_3 x^6}{6!} + \cdots \qquad |x| < \pi$$

## DEFINITION OF EULER NUMBERS

The *Euler numbers* $E_1, E_2, E_3, \ldots$ are defined by the series

**21.3** $$\operatorname{sech} x = 1 - \frac{E_1 x^2}{2!} + \frac{E_2 x^4}{4!} - \frac{E_3 x^6}{6!} + \cdots \qquad |x| < \frac{\pi}{2}$$

**21.4** $$\sec x = 1 + \frac{E_1 x^2}{2!} + \frac{E_2 x^4}{4!} + \frac{E_3 x^6}{6!} + \cdots \qquad |x| < \frac{\pi}{2}$$

## TABLE OF FIRST FEW BERNOULLI AND EULER NUMBERS

| Bernoulli numbers | Euler numbers |
|---|---|
| $B_1 = 1/6$ | $E_1 = 1$ |
| $B_2 = 1/30$ | $E_2 = 5$ |
| $B_3 = 1/42$ | $E_3 = 61$ |
| $B_4 = 1/30$ | $E_4 = 1385$ |
| $B_5 = 5/66$ | $E_5 = 50{,}521$ |
| $B_6 = 691/2730$ | $E_6 = 2{,}702{,}765$ |
| $B_7 = 7/6$ | $E_7 = 199{,}360{,}981$ |
| $B_8 = 3617/510$ | $E_8 = 19{,}391{,}512{,}145$ |
| $B_9 = 43{,}867/798$ | $E_9 = 2{,}404{,}879{,}675{,}441$ |
| $B_{10} = 174{,}611/330$ | $E_{10} = 370{,}371{,}188{,}237{,}525$ |
| $B_{11} = 854{,}513/138$ | $E_{11} = 69{,}348{,}874{,}393{,}137{,}901$ |
| $B_{12} = 236{,}364{,}091/2730$ | $E_{12} = 15{,}514{,}534{,}163{,}557{,}086{,}905$ |

## RELATIONSHIPS OF BERNOULLI AND EULER NUMBERS

**21.5** $$\binom{2n+1}{2}2^2B_1 - \binom{2n+1}{4}2^4B_2 + \binom{2n+1}{6}2^6B_3 - \cdots (-1)^{n-1}(2n+1)2^{2n}B_n \;=\; 2n$$

**21.6** $$E_n \;=\; \binom{2n}{2}E_{n-1} - \binom{2n}{4}E_{n-2} + \binom{2n}{6}E_{n-3} - \cdots (-1)^n$$

**21.7** $$B_n \;=\; \frac{2n}{2^{2n}(2^{2n}-1)}\left\{\binom{2n-1}{1}E_{n-1} - \binom{2n-1}{3}E_{n-2} + \binom{2n-1}{5}E_{n-3} - \cdots (-1)^{n-1}\right\}$$

## SERIES INVOLVING BERNOULLI AND EULER NUMBERS

**21.8** $$\overset{B_{2n}}{B_n} \;=\; \frac{(2n)!}{2^{2n-1}\pi^{2n}}\left\{1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots\right\} \qquad n \gg 1$$

**21.9** $$\overset{B_{2n}}{B_n} \;=\; \frac{2(2n)!}{(2^{2n}-1)\pi^{2n}}\left\{1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \cdots\right\} \qquad n \gg 1$$

**21.10** $$\overset{B_{2n}}{B_n} \;=\; \frac{(2n)!}{(2^{2n-1}-1)\pi^{2n}}\left\{1 - \frac{1}{2^{2n}} + \frac{1}{3^{2n}} - \cdots\right\} \qquad n \gg 1$$

**21.11** $$E_n \;=\; \frac{2^{2n+2}(2n)!}{\pi^{2n+1}}\left\{1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \cdots\right\} \qquad n \gg 1$$

## ASYMPTOTIC FORMULA FOR BERNOULLI NUMBERS

**21.12** $$\boxed{B_n \;\sim\; 4n^{2n}(\pi e)^{-2n}\sqrt{\pi n}}$$

# 22

# FORMULAS from VECTOR ANALYSIS

## VECTORS AND SCALARS

Various quantities in physics such as temperature, volume and speed can be specified by a real number. Such quantities are called *scalars.*

Other quantities such as force, velocity and momentum require for their specification a direction as well as magnitude. Such quantities are called *vectors.* A vector is represented by an arrow or directed line segment indicating direction. The magnitude of the vector is determined by the length of the arrow, using an appropriate unit.

## NOTATION FOR VECTORS

A vector is denoted by a bold faced letter such as **A** [Fig. 22-1]. The magnitude is denoted by $|\mathbf{A}|$ or $A$. The tail end of the arrow is called the *initial point* while the head is called the *terminal point.*

## FUNDAMENTAL DEFINITIONS

1. **Equality of vectors.** Two vectors are equal if they have the same magnitude and direction. Thus $\mathbf{A} = \mathbf{B}$ in Fig. 22-1.

2. **Multiplication of a vector by a scalar.** If $m$ is any real number (scalar), then $m\mathbf{A}$ is a vector whose magnitude is $|m|$ times the magnitude of **A** and whose direction is the same as or opposite to **A** according as $m > 0$ or $m < 0$. If $m = 0$, then $m\mathbf{A} = \mathbf{0}$ is called the *zero* or *null vector.*

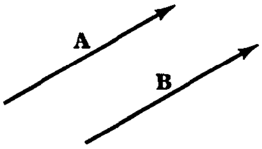
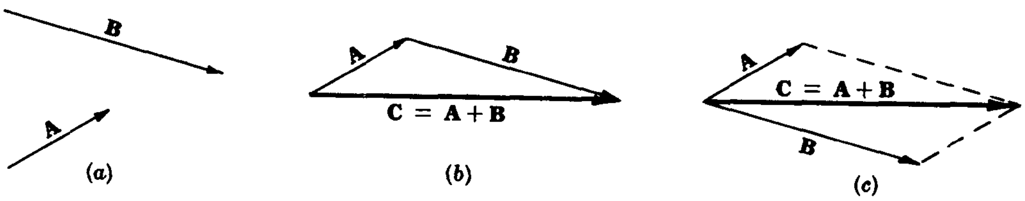


Fig. 22-1

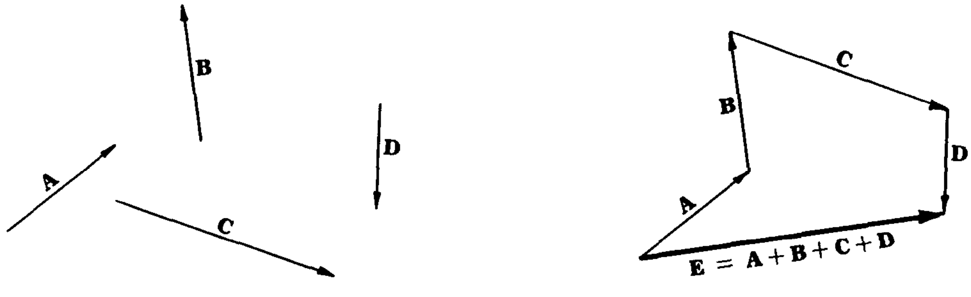
3. **Sums of vectors.** The sum or resultant of **A** and **B** is a vector $\mathbf{C} = \mathbf{A} + \mathbf{B}$ formed by placing the initial point of **B** on the terminal point of **A** and joining the initial point of **A** to the terminal point of **B** [Fig. 22-2(*b*)]. This definition is equivalent to the parallelogram law for vector addition as indicated in Fig. 22-2(*c*). The vector $\mathbf{A} - \mathbf{B}$ is defined as $\mathbf{A} + (-\mathbf{B})$.



(*a*) (*b*) (*c*)

Fig. 22-2

Extensions to sums of more than two vectors are immediate. Thus Fig. 22-3 shows how to obtain the sum **E** of the vectors **A**, **B**, **C** and **D**.



(*a*) (*b*)

Fig. 22-3

4. **Unit vectors.** A *unit vector* is a vector with unit magnitude. If **A** is a vector, then a unit vector in the direction of **A** is $\mathbf{a} = \mathbf{A}/A$ where $A > 0$.

## LAWS OF VECTOR ALGEBRA

If **A**, **B**, **C** are vectors and $m, n$ are scalars, then

**22.1** $\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$ Commutative law for addition

**22.2** $\mathbf{A}+(\mathbf{B}+\mathbf{C}) = (\mathbf{A}+\mathbf{B})+\mathbf{C}$ Associative law for addition

**22.3** $m(n\mathbf{A}) = (mn)\mathbf{A} = n(m\mathbf{A})$ Associative law for scalar multiplication

**22.4** $(m+n)\mathbf{A} = m\mathbf{A}+n\mathbf{A}$ Distributive law

**22.5** $m(\mathbf{A}+\mathbf{B}) = m\mathbf{A}+m\mathbf{B}$ Distributive law

## COMPONENTS OF A VECTOR

A vector **A** can be represented with initial point at the origin of a rectangular coordinate system. If **i**, **j**, **k** are unit vectors in the directions of the positive $x, y, z$ axes, then

**22.6** $$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$$

where $A_1\mathbf{i}, A_2\mathbf{j}, A_3\mathbf{k}$ are called *component vectors* of **A** in the **i**, **j**, **k** directions and $A_1, A_2, A_3$ are called the *components* of **A**.

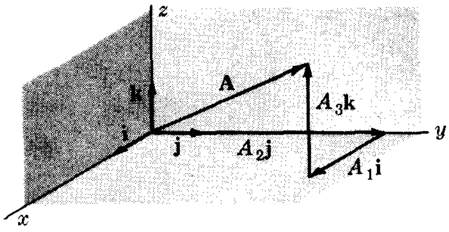


Fig. 22-4

## DOT OR SCALAR PRODUCT

**22.7** $$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta \qquad 0 \leqq \theta \leqq \pi$$

where $\theta$ is the angle between **A** and **B**.

Fundamental results are

**22.8** $$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$$ Commutative law

**22.9** $$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}$$ Distributive law

**22.10** $$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$$

where $\mathbf{A} = A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k}$, $\mathbf{B} = B_1\mathbf{i}+B_2\mathbf{j}+B_3\mathbf{k}$.

## CROSS OR VECTOR PRODUCT

**22.11** $$\mathbf{A}\times\mathbf{B} = AB\sin\theta\,\mathbf{u} \qquad 0 \leqq \theta \leqq \pi$$

where $\theta$ is the angle between **A** and **B** and **u** is a unit vector perpendicular to the plane of **A** and **B** such that **A**, **B**, **u** form a *right-handed system* [i.e. a right-threaded screw rotated through an angle less than 180° from **A** to **B** will advance in the direction of **u** as in Fig. 22-5].

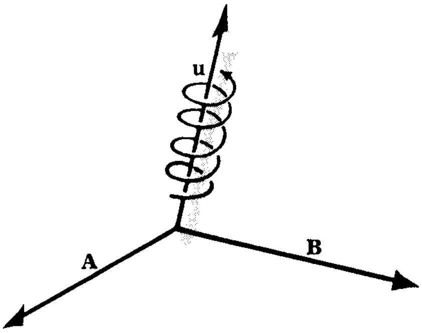


**Fig. 22-5**

Fundamental results are

**22.12** $$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$
$$= (A_2B_3 - A_3B_2)\mathbf{i} + (A_3B_1 - A_1B_3)\mathbf{j} + (A_1B_2 - A_2B_1)\mathbf{k}$$

**22.13** $$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$$

**22.14** $$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$$

**22.15** $|\mathbf{A}\times\mathbf{B}|$ = area of parallelogram having sides **A** and **B**

## MISCELLANEOUS FORMULAS INVOLVING DOT AND CROSS PRODUCTS

**22.16** $$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = A_1B_2C_3 + A_2B_3C_1 + A_3B_1C_2 - A_3B_2C_1 - A_2B_1C_3 - A_1B_3C_2$$

**22.17** $|\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})|$ = volume of parallelepiped with sides **A**, **B**, **C**

**22.18** $$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

**22.19** $$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{C})$$

**22.20** $$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$$

**22.21** $$(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = \mathbf{C}\{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{D})\} - \mathbf{D}\{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})\}$$
$$= \mathbf{B}\{\mathbf{A}\cdot(\mathbf{C}\times\mathbf{D})\} - \mathbf{A}\{\mathbf{B}\cdot(\mathbf{C}\times\mathbf{D})\}$$

## DERIVATIVES OF VECTORS

The derivative of a vector function $\mathbf{A}(u) = A_1(u)\mathbf{i} + A_2(u)\mathbf{j} + A_3(u)\mathbf{k}$ of the scalar variable $u$ is given by

**22.22** $$\frac{d\mathbf{A}}{du} = \lim_{\Delta u \to 0} \frac{\mathbf{A}(u+\Delta u) - \mathbf{A}(u)}{\Delta u} = \frac{dA_1}{du}\mathbf{i} + \frac{dA_2}{du}\mathbf{j} + \frac{dA_3}{du}\mathbf{k}$$

Partial derivatives of a vector function $\mathbf{A}(x,y,z)$ are similarly defined. We assume that all derivatives exist unless otherwise specified.

## FORMULAS INVOLVING DERIVATIVES

**22.23** $$\frac{d}{du}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du}\cdot\mathbf{B}$$

**22.24** $$\frac{d}{du}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du}\times\mathbf{B}$$

**22.25** $$\frac{d}{du}\{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})\} = \frac{d\mathbf{A}}{du}\cdot(\mathbf{B}\times\mathbf{C}) + \mathbf{A}\cdot\left(\frac{d\mathbf{B}}{du}\times\mathbf{C}\right) + \mathbf{A}\cdot\left(\mathbf{B}\times\frac{d\mathbf{C}}{du}\right)$$

**22.26** $$\mathbf{A}\cdot\frac{d\mathbf{A}}{du} = A\,\frac{dA}{du}$$

**22.27** $$\mathbf{A}\cdot\frac{d\mathbf{A}}{du} = 0 \quad \text{if } |\mathbf{A}| \text{ is a constant}$$

## THE DEL OPERATOR

The operator *del* is defined by

**22.28** $$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

In the results below we assume that $U = U(x,y,z)$, $V = V(x,y,z)$, $\mathbf{A} = \mathbf{A}(x,y,z)$ and $\mathbf{B} = \mathbf{B}(x,y,z)$ have partial derivatives.

## THE GRADIENT

**22.29** $$\text{Gradient of } U = \text{grad } U = \nabla U = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)U = \frac{\partial U}{\partial x}\mathbf{i} + \frac{\partial U}{\partial y}\mathbf{j} + \frac{\partial U}{\partial z}\mathbf{k}$$

## THE DIVERGENCE

**22.30** $$\begin{aligned}\text{Divergence of } \mathbf{A} = \text{div } \mathbf{A} = \nabla\cdot\mathbf{A} &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}) \\ &= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\end{aligned}$$

## THE CURL

**22.31** Curl of **A** $= \text{curl}\,\mathbf{A} = \nabla\times\mathbf{A}$

$$= \left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times(A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\mathbf{k}$$

## THE LAPLACIAN

**22.32** Laplacian of $U = \nabla^2 U = \nabla\cdot(\nabla U) = \dfrac{\partial^2 U}{\partial x^2}+\dfrac{\partial^2 U}{\partial y^2}+\dfrac{\partial^2 U}{\partial z^2}$

**22.33** Laplacian of **A** $= \nabla^2\mathbf{A} = \dfrac{\partial^2\mathbf{A}}{\partial x^2}+\dfrac{\partial^2\mathbf{A}}{\partial y^2}+\dfrac{\partial^2\mathbf{A}}{\partial z^2}$

## THE BIHARMONIC OPERATOR

**22.34** Biharmonic operator on $U = \nabla^4 U = \nabla^2(\nabla^2 U)$

$$= \frac{\partial^4 U}{\partial x^4}+\frac{\partial^4 U}{\partial y^4}+\frac{\partial^4 U}{\partial z^4}+2\frac{\partial^4 U}{\partial x^2\,\partial y^2}+2\frac{\partial^4 U}{\partial y^2\,\partial z^2}+2\frac{\partial^4 U}{\partial x^2\,\partial z^2}$$

## MISCELLANEOUS FORMULAS INVOLVING $\nabla$

**22.35** $\nabla(U+V) = \nabla U + \nabla V$

**22.36** $\nabla\cdot(\mathbf{A}+\mathbf{B}) = \nabla\cdot\mathbf{A} + \nabla\cdot\mathbf{B}$

**22.37** $\nabla\times(\mathbf{A}+\mathbf{B}) = \nabla\times\mathbf{A} + \nabla\times\mathbf{B}$

**22.38** $\nabla\cdot(U\mathbf{A}) = (\nabla U)\cdot\mathbf{A} + U(\nabla\cdot\mathbf{A})$

**22.39** $\nabla\times(U\mathbf{A}) = (\nabla U)\times\mathbf{A} + U(\nabla\times\mathbf{A})$

**22.40** $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

**22.41** $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - \mathbf{B}(\nabla\cdot\mathbf{A}) - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B})$

**22.42** $\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{B})$

**22.43** $\nabla\times(\nabla U) = \mathbf{0}$, i.e. the curl of the gradient of $U$ is zero.

**22.44** $\nabla\cdot(\nabla\times\mathbf{A}) = 0$, i.e. the divergence of the curl of **A** is zero.

**22.45** $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

## INTEGRALS INVOLVING VECTORS

If $\mathbf{A}(u) = \frac{d}{du}\mathbf{B}(u)$, then the *indefinite integral* of $\mathbf{A}(u)$ is

**22.46** $$\int \mathbf{A}(u)\,du = \mathbf{B}(u) + \mathbf{c} \qquad \mathbf{c} = \text{constant vector}$$

The *definite integral* of $\mathbf{A}(u)$ from $u = a$ to $u = b$ in this case is given by

**22.47** $$\int_a^b \mathbf{A}(u)\,du = \mathbf{B}(b) - \mathbf{B}(a)$$

The definite integral can be defined as on page 94.

## LINE INTEGRALS

Consider a space curve $C$ joining two points $P_1(a_1, a_2, a_3)$ and $P_2(b_1, b_2, b_3)$ as in Fig. 22-6. Divide the curve into $n$ parts by points of subdivision $(x_1, y_1, z_1), \ldots, (x_{n-1}, y_{n-1}, z_{n-1})$. Then the *line integral* of a vector $\mathbf{A}(x, y, z)$ along $C$ is defined as

**22.48** $$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_{P_1}^{P_2} \mathbf{A}\cdot d\mathbf{r} = \lim_{n\to\infty} \sum_{p=1}^{n} \mathbf{A}(x_p, y_p, z_p)\cdot \Delta\mathbf{r}_p$$

where $\Delta\mathbf{r}_p = \Delta x_p\,\mathbf{i} + \Delta y_p\,\mathbf{j} + \Delta z_p\,\mathbf{k}$, $\Delta x_p = x_{p+1} - x_p$, $\Delta y_p = y_{p+1} - y_p$, $\Delta z_p = z_{p+1} - z_p$ and where it is assumed that as $n \to \infty$ the largest of the magnitudes $|\Delta\mathbf{r}_p|$ approaches zero. The result 22.48 is a generalization of the ordinary definite integral [page 94].

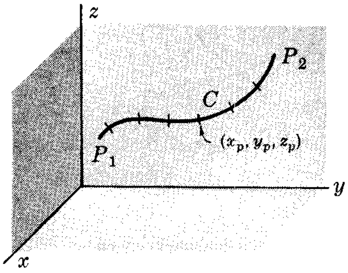


Fig. 22-6

The line integral 22.48 can also be written

**22.49** $$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_C (A_1\,dx + A_2\,dy + A_3\,dz)$$

using $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$.

## PROPERTIES OF LINE INTEGRALS

**22.50** $$\int_{P_1}^{P_2} \mathbf{A}\cdot d\mathbf{r} = -\int_{P_2}^{P_1} \mathbf{A}\cdot d\mathbf{r}$$

**22.51** $$\int_{P_1}^{P_2} \mathbf{A}\cdot d\mathbf{r} = \int_{P_1}^{P_3} \mathbf{A}\cdot d\mathbf{r} + \int_{P_3}^{P_2} \mathbf{A}\cdot d\mathbf{r}$$

## INDEPENDENCE OF THE PATH

In general a line integral has a value which depends on the particular path $C$ joining points $P_1$ and $P_2$ in a region $\mathcal{R}$. However, in case $\mathbf{A} = \nabla\phi$ or $\nabla \times \mathbf{A} = \mathbf{0}$ where $\phi$ and its partial derivatives are continuous in $\mathcal{R}$, the line integral $\int_C \mathbf{A}\cdot d\mathbf{r}$ is independent of the path. In such case

**22.52** $$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_{P_1}^{P_2} \mathbf{A}\cdot d\mathbf{r} = \phi(P_2) - \phi(P_1)$$

where $\phi(P_1)$ and $\phi(P_2)$ denote the values of $\phi$ at $P_1$ and $P_2$ respectively. In particular if $C$ is a closed curve,

**22.53**
$$\int_C \mathbf{A}\cdot d\mathbf{r} \;=\; \oint_C \mathbf{A}\cdot d\mathbf{r} \;=\; 0$$

where the circle on the integral sign is used to emphasize that $C$ is closed.

## MULTIPLE INTEGRALS

Let $F(x,y)$ be a function defined in a region $\mathcal{R}$ of the $xy$ plane as in Fig. 22-7. Subdivide the region into $n$ parts by lines parallel to the $x$ and $y$ axes as indicated. Let $\Delta A_p = \Delta x_p\,\Delta y_p$ denote an area of one of these parts. Then the integral of $F(x,y)$ over $\mathcal{R}$ is defined as

**22.54**
$$\int_{\mathcal{R}} F(x,y)\,dA \;=\; \lim_{n\to\infty}\sum_{p=1}^{n} F(x_p,y_p)\,\Delta A_p$$

provided this limit exists.

In such case the integral can also be written as

**22.55**
$$\int_{x=a}^{b}\int_{y=f_1(x)}^{f_2(x)} F(x,y)\,dy\,dx$$
$$= \int_{x=a}^{b}\left\{\int_{y=f_1(x)}^{f_2(x)} F(x,y)\,dy\right\}dx$$

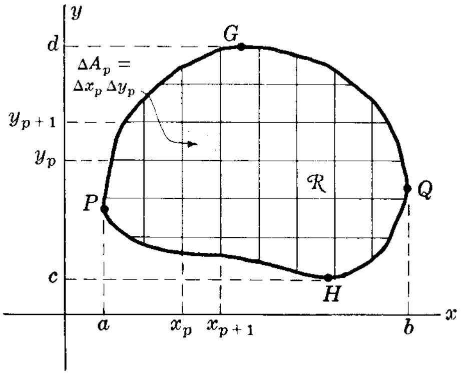


Fig. 22-7

where $y = f_1(x)$ and $y = f_2(x)$ are the equations of curves $PHQ$ and $PGQ$ respectively and $a$ and $b$ are the $x$ coordinates of points $P$ and $Q$. The result can also be written as

**22.56**
$$\int_{y=c}^{d}\int_{x=g_1(y)}^{g_2(y)} F(x,y)\,dx\,dy \;=\; \int_{y=c}^{d}\left\{\int_{x=g_1(y)}^{g_2(y)} F(x,y)\,dx\right\}dy$$

where $x = g_1(y)$, $x = g_2(y)$ are the equations of curves $HPG$ and $HQG$ respectively and $c$ and $d$ are the $y$ coordinates of $H$ and $G$.

These are called *double integrals* or *area integrals*. The ideas can be similarly extended to *triple* or *volume integrals* or to higher *multiple integrals.*

## SURFACE INTEGRALS

Subdivide the surface $S$ [see Fig. 22-8] into $n$ elements of area $\Delta S_p$, $p = 1,2,\ldots,n$. Let $\mathbf{A}(x_p,y_p,z_p) = \mathbf{A}_p$ where $(x_p,y_p,z_p)$ is a point $P$ in $\Delta S_p$. Let $\mathbf{N}_p$ be a unit normal to $\Delta S_p$ at $P$. Then the surface integral of the normal component of $\mathbf{A}$ over $S$ is defined as

**22.57**
$$\int_S \mathbf{A}\cdot\mathbf{N}\,dS \;=\; \lim_{n\to\infty}\sum_{p=1}^{n}\mathbf{A}_p\cdot\mathbf{N}_p\,\Delta S_p$$

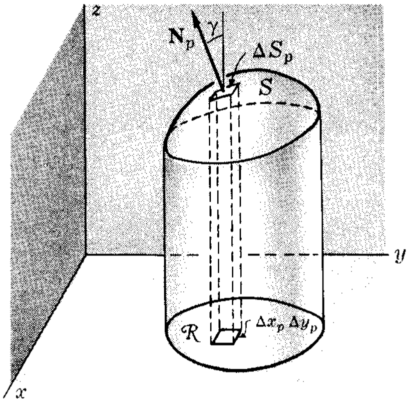


Fig. 22-8

## RELATION BETWEEN SURFACE AND DOUBLE INTEGRALS

If $\mathcal{R}$ is the projection of $S$ on the $xy$ plane, then [see Fig. 22-8]

**22.58**
$$\int_S \mathbf{A}\cdot\mathbf{N}\,dS = \iint_{\mathcal{R}} \mathbf{A}\cdot\mathbf{N}\,\frac{dx\,dy}{|\mathbf{N}\cdot\mathbf{k}|}$$

## THE DIVERGENCE THEOREM

Let $S$ be a closed surface bounding a region of volume $V$; then if $\mathbf{N}$ is the positive (outward drawn) normal and $d\mathbf{S} = \mathbf{N}\,dS$, we have [see Fig. 22-9]

**22.59**
$$\int_V \nabla\cdot\mathbf{A}\,dV = \int_S \mathbf{A}\cdot d\mathbf{S}$$

The result is also called *Gauss' theorem* or *Green's theorem.*

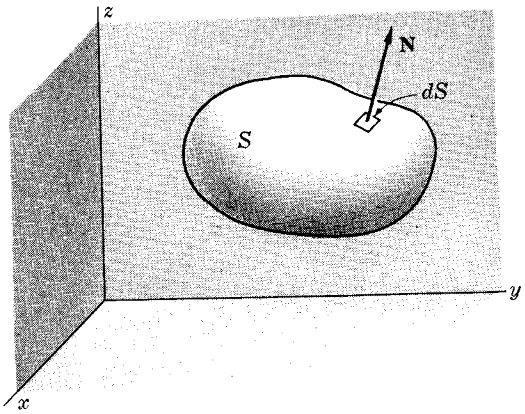


Fig. 22-9

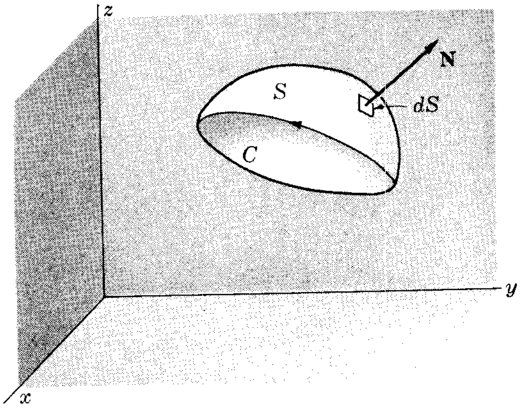


Fig. 22-10

## STOKE'S THEOREM

Let $S$ be an open two-sided surface bounded by a closed non-intersecting curve $C$ [simple closed curve] as in Fig. 22-10. Then

**22.60**
$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \int_S (\nabla\times\mathbf{A})\cdot d\mathbf{S}$$

where the circle on the integral is used to emphasize that $C$ is closed.

## GREEN'S THEOREM IN THE PLANE

**22.61**
$$\oint_C (P\,dx + Q\,dy) = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

where $R$ is the area bounded by the closed curve $C$. This result is a special case of the divergence theorem or Stoke's theorem.

## GREEN'S FIRST IDENTITY

**22.62**
$$\int_V \{\phi\,\nabla^2\psi + (\nabla\phi)\cdot(\nabla\psi)\}\,dV \;=\; \int (\phi\,\nabla\psi)\cdot d\mathbf{S}$$

where $\phi$ and $\psi$ are scalar functions.

## GREEN'S SECOND IDENTITY

**22.63**
$$\int_V (\phi\,\nabla^2\psi - \psi\,\nabla^2\phi)\,dV \;=\; \int_S (\phi\,\nabla\psi - \psi\nabla\phi)\cdot d\mathbf{S}$$

## MISCELLANEOUS INTEGRAL THEOREMS

**22.64** $\displaystyle\int_V \nabla\times\mathbf{A}\,dV \;=\; \int_S d\mathbf{S}\times\mathbf{A}$ **22.65** $\displaystyle\int_C \phi\,d\mathbf{r} \;=\; \int_S d\mathbf{S}\times\nabla\phi$

## CURVILINEAR COORDINATES

A point $P$ in space [see Fig. 22-11] can be located by rectangular coordinates $(x, y, z)$ or curvilinear coordinates $(u_1, u_2, u_3)$ where the transformation equations from one set of coordinates to the other are given by

**22.66**
$$x = x(u_1, u_2, u_3)$$
$$y = y(u_1, u_2, u_3)$$
$$z = z(u_1, u_2, u_3)$$

If $u_2$ and $u_3$ are constant, then as $u_1$ varies, the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ of $P$ describes a curve called the $u_1$ coordinate curve. Similarly we define the $u_2$ and $u_3$ coordinate curves through $P$. The vectors $\partial\mathbf{r}/\partial u_1$, $\partial\mathbf{r}/\partial u_2$, $\partial\mathbf{r}/\partial u_3$ represent tangent vectors to the $u_1, u_2, u_3$ coordinate curves. Letting $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be unit tangent vectors to these curves, we have

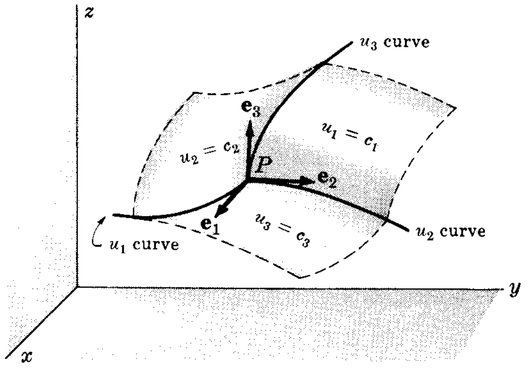


Fig. 22-11

**22.67**
$$\frac{\partial\mathbf{r}}{\partial u_1} = h_1\mathbf{e}_1, \quad \frac{\partial\mathbf{r}}{\partial u_2} = h_2\mathbf{e}_2, \quad \frac{\partial\mathbf{r}}{\partial u_3} = h_3\mathbf{e}_3$$

where

**22.68**
$$h_1 = \left|\frac{\partial\mathbf{r}}{\partial u_1}\right|, \quad h_2 = \left|\frac{\partial\mathbf{r}}{\partial u_2}\right|, \quad h_3 = \left|\frac{\partial\mathbf{r}}{\partial u_3}\right|$$

are called *scale factors.* If $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are mutually perpendicular, the curvilinear coordinate system is called *orthogonal.*

## FORMULAS INVOLVING ORTHOGONAL CURVILINEAR COORDINATES

**22.69**
$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1}du_1 + \frac{\partial \mathbf{r}}{\partial u_2}du_2 + \frac{\partial \mathbf{r}}{\partial u_3}du_3 = h_1\,du_1\,\mathbf{e}_1 + h_2\,du_2\,\mathbf{e}_2 + h_3\,du_3\,\mathbf{e}_3$$

**22.70**
$$ds^2 = d\mathbf{r}\cdot d\mathbf{r} = h_1^2\,du_1^2 + h_2^2\,du_2^2 + h_3^2\,du_3^2$$

where $ds$ is the element of arc length.

If $dV$ is the element of volume, then

**22.71**
$$dV = |(h_1\mathbf{e}_1\,du_1)\cdot(h_2\mathbf{e}_2\,du_2)\times(h_3\mathbf{e}_3\,du_3)| = h_1h_2h_3\,du_1\,du_2\,du_3$$
$$= \left|\frac{\partial \mathbf{r}}{\partial u_1}\cdot\frac{\partial \mathbf{r}}{\partial u_2}\times\frac{\partial \mathbf{r}}{\partial u_3}\right| du_1\,du_2\,du_3 = \left|\frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)}\right| du_1\,du_2\,du_3$$

where

**22.72**
$$\frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)} = \begin{vmatrix} \partial x/\partial u_1 & \partial x/\partial u_2 & \partial x/\partial u_3 \\ \partial y/\partial u_1 & \partial y/\partial u_2 & \partial y/\partial u_3 \\ \partial z/\partial u_1 & \partial z/\partial u_2 & \partial z/\partial u_3 \end{vmatrix}$$

is called the *Jacobian* of the transformation.

## TRANSFORMATION OF MULTIPLE INTEGRALS

The result 22.72 can be used to transform multiple integrals from rectangular to curvilinear coordinates. For example, we have

**22.73**
$$\iiint_{\mathcal{R}} F(x,y,z)\,dx\,dy\,dz = \iiint_{\mathcal{R}'} G(u_1,u_2,u_3)\left|\frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)}\right| du_1\,du_2\,du_3$$

where $\mathcal{R}'$ is the region into which $\mathcal{R}$ is mapped by the transformation and $G(u_1,u_2,u_3)$ is the value of $F(x,y,z)$ corresponding to the transformation.

## GRADIENT, DIVERGENCE, CURL AND LAPLACIAN

In the following, $\Phi$ is a scalar function and $\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3$ a vector function of orthogonal curvilinear coordinates $u_1, u_2, u_3$.

**22.74** Gradient of $\Phi$ = grad $\Phi$ = $\nabla\Phi$ =
$$\frac{1}{h_1}\frac{\partial\Phi}{\partial u_1} + \frac{1}{h_2}\frac{\partial\Phi}{\partial u_2} + \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}$$

**22.75** Divergence of $\mathbf{A}$ = div $\mathbf{A}$ = $\nabla\cdot\mathbf{A}$ =
$$\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_3h_1A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$$

**22.76** Curl of $\mathbf{A}$ = curl $\mathbf{A}$ = $\nabla\times\mathbf{A}$ =
$$\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$
$$= \frac{1}{h_2h_3}\left[\frac{\partial}{\partial u_2}(h_3A_3) - \frac{\partial}{\partial u_3}(h_2A_2)\right]\mathbf{e}_1 + \frac{1}{h_1h_3}\left[\frac{\partial}{\partial u_3}(h_1A_1) - \frac{\partial}{\partial u_1}(h_3A_3)\right]\mathbf{e}_2$$
$$+ \frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2A_2) - \frac{\partial}{\partial u_2}(h_1A_1)\right]\mathbf{e}_3$$

**22.77** Laplacian of $\Phi$ = $\nabla^2\Phi$ =
$$\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\Phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3}\right)\right]$$

Note that the biharmonic operator $\nabla^4\Phi = \nabla^2(\nabla^2\Phi)$ can be obtained from 22.77.

## SPECIAL ORTHOGONAL COORDINATE SYSTEMS

### Cylindrical Coordinates $(r, \theta, z)$ [See Fig. 22-12]

**22.78** $$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

**22.79** $$h_1^2 = 1, \quad h_2^2 = r^2, \quad h_3^2 = 1$$

**22.80** $$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2}$$

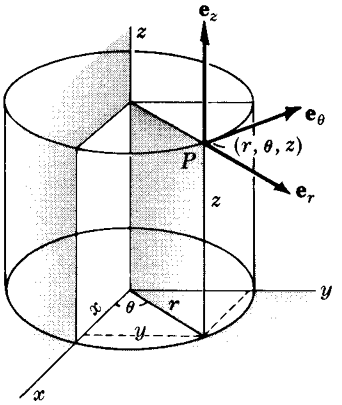

$\mathbf{e}_z$
$z$
$\mathbf{e}_\theta$
$(r, \theta, z)$
$P$
$z$
$\mathbf{e}_r$
$y$
$x$
$\theta$
$r$
$y$
$x$

Fig. 22-12. Cylindrical coordinates.

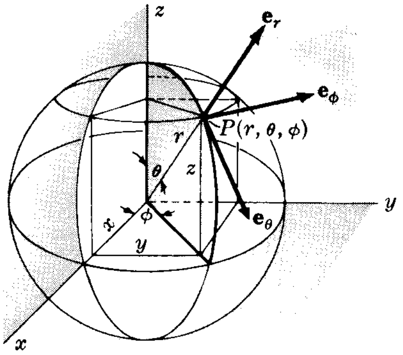

$z$
$\mathbf{e}_r$
$\mathbf{e}_\phi$
$P(r, \theta, \phi)$
$r$
$\theta$
$z$
$\phi$
$x$
$y$
$\mathbf{e}_\theta$
$y$
$x$

Fig. 22-13. Spherical coordinates.

### Spherical Coordinates $(r, \theta, \phi)$ [See Fig. 22-13]

**22.81** $$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

**22.82** $$h_1^2 = 1, \quad h_2^2 = r^2, \quad h_3^2 = r^2\sin^2\theta$$

**22.83** $$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

### Parabolic Cylindrical Coordinates $(u, v, z)$

**22.84** $$x = \tfrac{1}{2}(u^2 - v^2), \quad y = uv, \quad z = z$$

**22.85** $$h_1^2 = h_2^2 = u^2 + v^2, \quad h_3^2 = 1$$

**22.86** $$\nabla^2\Phi = \frac{1}{u^2 + v^2}\left(\frac{\partial^2\Phi}{\partial u^2} + \frac{\partial^2\Phi}{\partial v^2}\right) + \frac{\partial^2\Phi}{\partial z^2}$$

The traces of the coordinate surfaces on the $xy$ plane are shown in Fig. 22-14. They are confocal parabolas with a common axis.

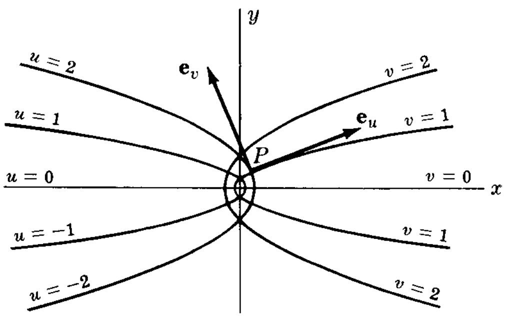

$y$
$u = 2$
$\mathbf{e}_v$
$v = 2$
$u = 1$
$\mathbf{e}_u$
$v = 1$
$P$
$u = 0$
$v = 0$
$x$
$u = -1$
$v = 1$
$u = -2$
$v = 2$

Fig. 22-14

## Paraboloidal Coordinates $(u, v, \phi)$

**22.87** $$x = uv\cos\phi, \quad y = uv\sin\phi, \quad z = \tfrac{1}{2}(u^2 - v^2)$$

where $$u \geqq 0, \quad v \geqq 0, \quad 0 \leqq \phi < 2\pi$$

**22.88** $$h_1^2 = h_2^2 = u^2 + v^2, \quad h_3^2 = u^2v^2$$

**22.89** $$\nabla^2\Phi = \frac{1}{u(u^2+v^2)}\frac{\partial}{\partial u}\left(u\frac{\partial\Phi}{\partial u}\right) + \frac{1}{v(u^2+v^2)}\frac{\partial}{\partial v}\left(v\frac{\partial\Phi}{\partial v}\right) + \frac{1}{u^2v^2}\frac{\partial^2\Phi}{\partial\phi^2}$$

Two sets of coordinate surfaces are obtained by revolving the parabolas of Fig. 22-14 about the $x$ axis which is then relabeled the $z$ axis.

## Elliptic Cylindrical Coordinates $(u, v, z)$

**22.90** $$x = a\cosh u\cos v, \quad y = a\sinh u\sin v, \quad z = z$$

where $$u \geqq 0, \quad 0 \leqq v < 2\pi, \quad -\infty < z < \infty$$

**22.91** $$h_1^2 = h_2^2 = a^2(\sinh^2 u + \sin^2 v), \quad h_3^2 = 1$$

**22.92** $$\nabla^2\Phi = \frac{1}{a^2(\sinh^2 u + \sin^2 v)}\left(\frac{\partial^2\Phi}{\partial u^2} + \frac{\partial^2\Phi}{\partial v^2}\right) + \frac{\partial^2\Phi}{\partial z^2}$$

The traces of the coordinate surfaces on the $xy$ plane are shown in Fig. 22-15. They are confocal ellipses and hyperbolas.

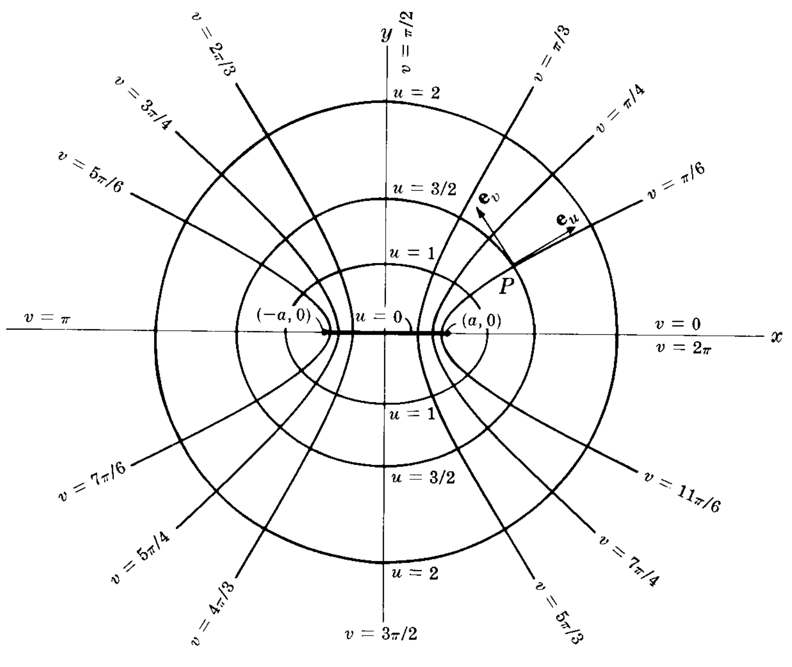


Fig. 22-15. Elliptic cylindrical coordinates.

### Prolate Spheroidal Coordinates $(\xi, \eta, \phi)$

**22.93** $$x = a \sinh \xi \sin \eta \cos \phi, \quad y = a \sinh \xi \sin \eta \sin \phi, \quad z = a \cosh \xi \cos \eta$$

where $$\xi \geqq 0, \quad 0 \leqq \eta \leqq \pi, \quad 0 \leqq \phi < 2\pi$$

**22.94** $$h_1^2 = h_2^2 = a^2(\sinh^2 \xi + \sin^2 \eta), \quad h_3^2 = a^2 \sinh^2 \xi \sin^2 \eta$$

**22.95** $$\nabla^2\Phi = \frac{1}{a^2(\sinh^2 \xi + \sin^2 \eta) \sinh \xi} \frac{\partial}{\partial \xi}\left(\sinh \xi \frac{\partial \Phi}{\partial \xi}\right) + \frac{1}{a^2(\sinh^2 \xi + \sin^2 \eta) \sin \eta} \frac{\partial}{\partial \eta}\left(\sin \eta \frac{\partial \Phi}{\partial \eta}\right) + \frac{1}{a^2 \sinh^2 \xi \sin^2 \eta} \frac{\partial^2 \Phi}{\partial \phi^2}$$

Two sets of coordinate surfaces are obtained by revolving the curves of Fig. 22-15 about the $x$ axis which is relabeled the $z$ axis. The third set of coordinate surfaces consists of planes passing through this axis.

### Oblate Spheroidal Coordinates $(\xi, \eta, \phi)$

**22.96** $$x = a \cosh \xi \cos \eta \cos \phi, \quad y = a \cosh \xi \cos \eta \sin \phi, \quad z = a \sinh \xi \sin \eta$$

where $$\xi \geqq 0, \quad -\pi/2 \leqq \eta \leqq \pi/2, \quad 0 \leqq \phi < 2\pi$$

**22.97** $$h_1^2 = h_2^2 = a^2(\sinh^2 \xi + \sin^2 \eta), \quad h_3^2 = a^2 \cosh^2 \xi \cos^2 \eta$$

**22.98** $$\nabla^2\Phi = \frac{1}{a^2(\sinh^2 \xi + \sin^2 \eta) \cosh \xi} \frac{\partial}{\partial \xi}\left(\cosh \xi \frac{\partial \Phi}{\partial \xi}\right) + \frac{1}{a^2(\sinh^2 \xi + \sin^2 \eta) \cos \eta} \frac{\partial}{\partial \eta}\left(\cos \eta \frac{\partial \Phi}{\partial \eta}\right) + \frac{1}{a^2 \cosh^2 \xi \cos^2 \eta} \frac{\partial^2 \Phi}{\partial \phi^2}$$

Two sets of coordinate surfaces are obtained by revolving the curves of Fig. 22-15 about the $y$ axis which is relabeled the $z$ axis. The third set of coordinate surfaces are planes passing through this axis.

### Bipolar Coordinates $(u, v, z)$

**22.99** $$x = \frac{a \sinh v}{\cosh v - \cos u}, \quad y = \frac{a \sin u}{\cosh v - \cos u}, \quad z = z$$

where $$0 \leqq u < 2\pi, \quad -\infty < v < \infty, \quad -\infty < z < \infty$$

or

**22.100** $$x^2 + (y - a \cot u)^2 = a^2 \csc^2 u, \quad (x - a \coth v)^2 + y^2 = a^2 \operatorname{csch}^2 v, \quad z = z$$

**22.101** $$h_1^2 = h_2^2 = \frac{a^2}{(\cosh v - \cos u)^2}, \quad h_3^2 = 1$$

**22.102** $$\nabla^2\Phi = \frac{(\cosh v - \cos u)^2}{a^2}\left(\frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2}\right) + \frac{\partial^2 \Phi}{\partial z^2}$$

The traces of the coordinate surfaces on the $xy$ plane are shown in Fig. 22-16 below.

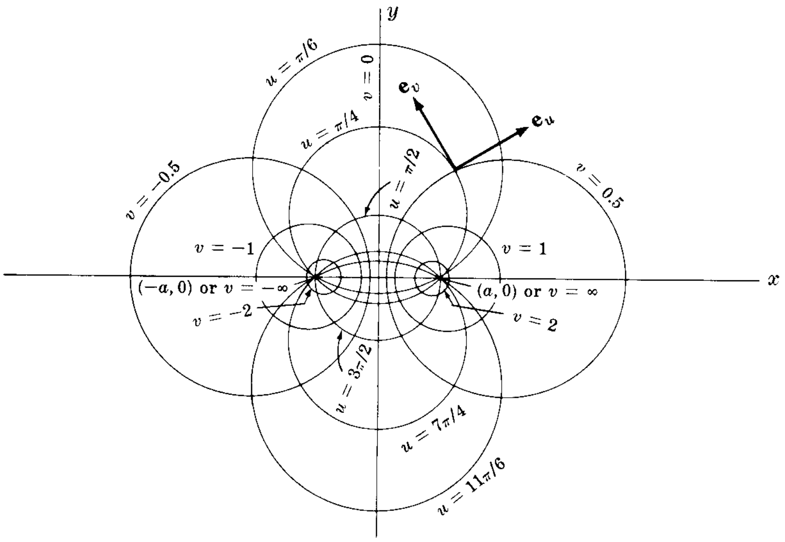


Fig. 22-16. Bipolar coordinates.

## Toroidal Coordinates $(u, v, \phi)$

**22.103** $$x = \frac{a \sinh v \cos\phi}{\cosh v - \cos u}, \quad y = \frac{a \sinh v \sin\phi}{\cosh v - \cos u}, \quad z = \frac{a \sin u}{\cosh v - \cos u}$$

**22.104** $$h_1^2 = h_2^2 = \frac{a^2}{(\cosh v - \cos u)^2}, \quad h_3^2 = \frac{a^2 \sinh^2 v}{(\cosh v - \cos u)^2}$$

**22.105** $$\nabla^2\Phi = \frac{(\cosh v - \cos u)^3}{a^2} \frac{\partial}{\partial u}\left(\frac{1}{\cosh v - \cos u} \frac{\partial\Phi}{\partial u}\right) + \frac{(\cosh v - \cos u)^3}{a^2 \sinh v} \frac{\partial}{\partial v}\left(\frac{\sinh v}{\cosh v - \cos u} \frac{\partial\Phi}{\partial v}\right) + \frac{(\cosh v - \cos u)^2}{a^2 \sinh^2 v} \frac{\partial^2\Phi}{\partial\phi^2}$$

The coordinate surfaces are obtained by revolving the curves of Fig. 22-16 about the $y$ axis which is relabeled the $z$ axis.

## Conical Coordinates $(\lambda, \mu, \nu)$

**22.106** $$x = \frac{\lambda\mu\nu}{ab}, \quad y = \frac{\lambda}{a}\sqrt{\frac{(\mu^2 - a^2)(\nu^2 - a^2)}{a^2 - b^2}}, \quad z = \frac{\lambda}{b}\sqrt{\frac{(\mu^2 - b^2)(\nu^2 - b^2)}{b^2 - a^2}}$$

**22.107** $$h_1^2 = 1, \quad h_2^2 = \frac{\lambda^2(\mu^2 - \nu^2)}{(\mu^2 - a^2)(b^2 - \mu^2)}, \quad h_3^2 = \frac{\lambda^2(\mu^2 - \nu^2)}{(\nu^2 - a^2)(\nu^2 - b^2)}$$

## Confocal Ellipsoidal Coordinates $(\lambda, \mu, \nu)$

**22.108**
$$\begin{cases} \dfrac{x^2}{a^2-\lambda} + \dfrac{y^2}{b^2-\lambda} + \dfrac{z^2}{c^2-\lambda} = 1 & \lambda < c^2 < b^2 < a^2 \\ \dfrac{x^2}{a^2-\mu} + \dfrac{y^2}{b^2-\mu} + \dfrac{z^2}{c^2-\mu} = 1 & c^2 < \mu < b^2 < a^2 \\ \dfrac{x^2}{a^2-\nu} + \dfrac{y^2}{b^2-\nu} + \dfrac{z^2}{c^2-\nu} = 1 & c^2 < b^2 < \nu < a^2 \end{cases}$$

or

**22.109**
$$\begin{cases} x^2 = \dfrac{(a^2-\lambda)(a^2-\mu)(a^2-\nu)}{(a^2-b^2)(a^2-c^2)} \\ y^2 = \dfrac{(b^2-\lambda)(b^2-\mu)(b^2-\nu)}{(b^2-a^2)(b^2-c^2)} \\ z^2 = \dfrac{(c^2-\lambda)(c^2-\mu)(c^2-\nu)}{(c^2-a^2)(c^2-b^2)} \end{cases}$$

**22.110**
$$\begin{cases} h_1^2 = \dfrac{(\mu-\lambda)(\nu-\lambda)}{4(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)} \\ h_2^2 = \dfrac{(\nu-\mu)(\lambda-\mu)}{4(a^2-\mu)(b^2-\mu)(c^2-\mu)} \\ h_3^2 = \dfrac{(\lambda-\nu)(\mu-\nu)}{4(a^2-\nu)(b^2-\nu)(c^2-\nu)} \end{cases}$$

## Confocal Paraboloidal Coordinates $(\lambda, \mu, \nu)$

**22.111**
$$\begin{cases} \dfrac{x^2}{a^2-\lambda} + \dfrac{y^2}{b^2-\lambda} = z - \lambda & -\infty < \lambda < b^2 \\ \dfrac{x^2}{a^2-\mu} + \dfrac{y^2}{b^2-\mu} = z - \mu & b^2 < \mu < a^2 \\ \dfrac{x^2}{a^2-\nu} + \dfrac{y^2}{b^2-\nu} = z - \nu & a^2 < \nu < \infty \end{cases}$$

or

**22.112**
$$\begin{cases} x^2 = \dfrac{(a^2-\lambda)(a^2-\mu)(a^2-\nu)}{b^2-a^2} \\ y^2 = \dfrac{(b^2-\lambda)(b^2-\mu)(b^2-\nu)}{a^2-b^2} \\ z = \lambda + \mu + \nu - a^2 - b^2 \end{cases}$$

**22.113**
$$\begin{cases} h_1^2 = \dfrac{(\mu-\lambda)(\nu-\lambda)}{4(a^2-\lambda)(b^2-\lambda)} \\ h_2^2 = \dfrac{(\nu-\mu)(\lambda-\mu)}{4(a^2-\mu)(b^2-\mu)} \\ h_3^2 = \dfrac{(\lambda-\nu)(\mu-\nu)}{16(a^2-\nu)(b^2-\nu)} \end{cases}$$

# 23 FOURIER SERIES

## DEFINITION OF A FOURIER SERIES

The Fourier series corresponding to a function $f(x)$ defined in the interval $c \leqq x \leqq c+2L$ where $c$ and $L > 0$ are constants, is defined as

**23.1**
$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where

**23.2**
$$\begin{cases} a_n = \dfrac{1}{L}\displaystyle\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}\,dx \\ b_n = \dfrac{1}{L}\displaystyle\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}\,dx \end{cases}$$

If $f(x)$ and $f'(x)$ are piecewise continuous and $f(x)$ is defined by periodic extension of period $2L$, i.e. $f(x+2L) = f(x)$, then the series converges to $f(x)$ if $x$ is a point of continuity and to $\frac{1}{2}\{f(x+0) + f(x-0)\}$ if $x$ is a point of discontinuity.

## COMPLEX FORM OF FOURIER SERIES

Assuming that the series 23.1 converges to $f(x)$, we have

**23.3**
$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

where

**23.4**
$$c_n = \frac{1}{L}\int_c^{c+2L} f(x)e^{-in\pi x/L}\,dx = \begin{cases} \frac{1}{2}(a_n - ib_n) & n > 0 \\ \frac{1}{2}(a_{-n} + ib_{-n}) & n < 0 \\ \frac{1}{2}a_0 & n = 0 \end{cases}$$

## PARSEVAL'S IDENTITY

**23.5**
$$\frac{1}{L}\int_c^{c+2L} \{f(x)\}^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

## GENERALIZED PARSEVAL IDENTITY

**23.6**
$$\frac{1}{L}\int_c^{c+2L} f(x)\,g(x)\,dx = \frac{a_0 c_0}{2} + \sum_{n=1}^{\infty}(a_n c_n + b_n d_n)$$

where $a_n, b_n$ and $c_n, d_n$ are the Fourier coefficients corresponding to $f(x)$ and $g(x)$ respectively.

## SPECIAL FOURIER SERIES AND THEIR GRAPHS

**23.7** $f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$

$$\frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$$

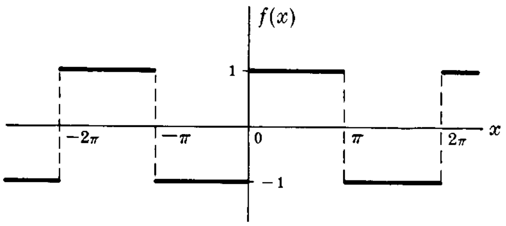


Fig. 23-1

**23.8** $f(x) = |x| = \begin{cases} x & 0 < x < \pi \\ -x & -\pi < x < 0 \end{cases}$

$$\frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$$

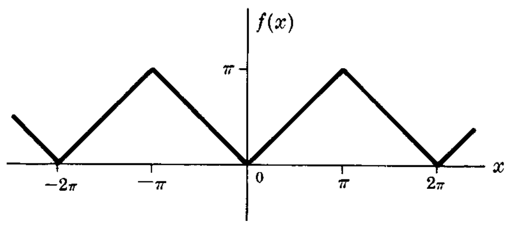


Fig. 23-2

**23.9** $f(x) = x, \ -\pi < x < \pi$

$$2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right)$$

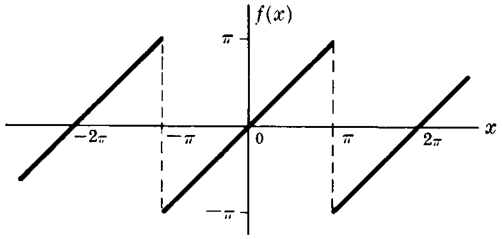


Fig. 23-3

**23.10** $f(x) = x, \ 0 < x < 2\pi$

$$\pi - 2\left(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right)$$

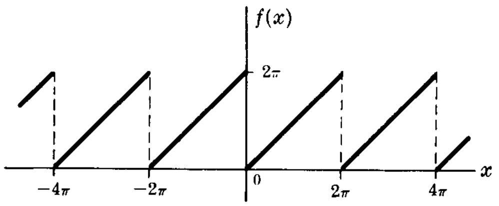


Fig. 23-4

**23.11** $f(x) = |\sin x|, \ -\pi < x < \pi$

$$\frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{1 \cdot 3} + \frac{\cos 4x}{3 \cdot 5} + \frac{\cos 6x}{5 \cdot 7} + \cdots\right)$$

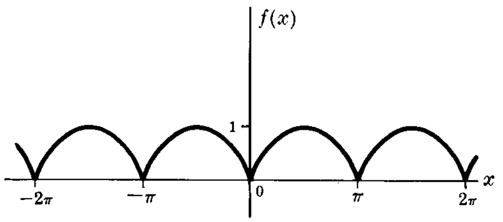


Fig. 23-5

**23.12** $f(x) = \begin{cases} \sin x & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases}$

$$\frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\left(\frac{\cos 2x}{1\cdot 3} + \frac{\cos 4x}{3\cdot 5} + \frac{\cos 6x}{5\cdot 7} + \cdots\right)$$

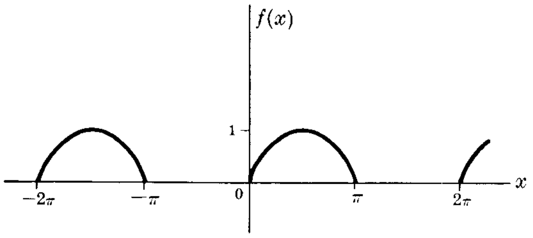


Fig. 23-6

**23.13** $f(x) = \begin{cases} \cos x & 0 < x < \pi \\ -\cos x & -\pi < x < 0 \end{cases}$

$$\frac{8}{\pi}\left(\frac{\sin 2x}{1\cdot 3} + \frac{2\sin 4x}{3\cdot 5} + \frac{3\sin 6x}{5\cdot 7} + \cdots\right)$$

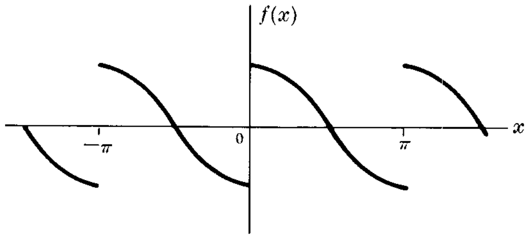


Fig. 23-7

**23.14** $f(x) = x^2, \quad -\pi < x < \pi$

$$\frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right)$$

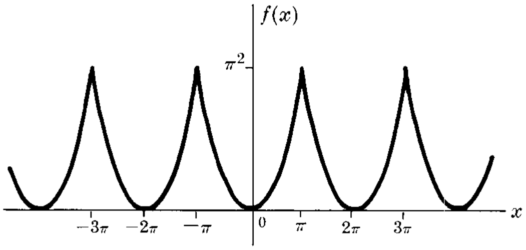


Fig. 23-8

**23.15** $f(x) = x(\pi - x), \quad 0 < x < \pi$

$$\frac{\pi^2}{6} - \left(\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \cdots\right)$$

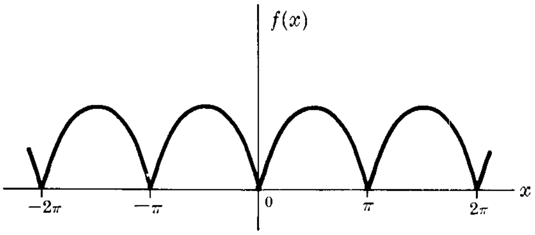


Fig. 23-9

**23.16** $f(x) = x(\pi - x)(\pi + x), \quad -\pi < x < \pi$

$$12\left(\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \cdots\right)$$

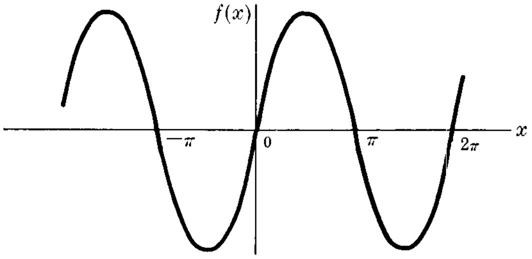


Fig. 23-10

**23.17** $$f(x) = \begin{cases} 0 & 0 < x < \pi - \alpha \\ 1 & \pi - \alpha < x < \pi + \alpha \\ 0 & \pi + \alpha < x < 2\pi \end{cases}$$

$$\frac{\alpha}{\pi} - \frac{2}{\pi}\left(\frac{\sin \alpha \cos x}{1} - \frac{\sin 2\alpha \cos 2x}{2} + \frac{\sin 3\alpha \cos 3x}{3} - \cdots\right)$$

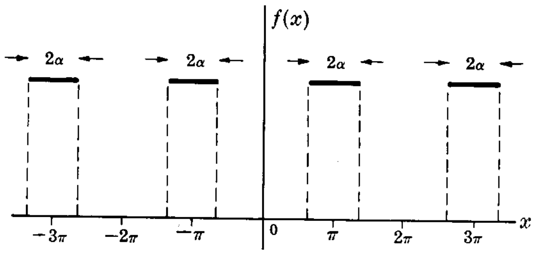


Fig. 23-11

**23.18** $$f(x) = \begin{cases} x(\pi - x) & 0 < x < \pi \\ -x(\pi - x) & -\pi < x < 0 \end{cases}$$

$$\frac{8}{\pi}\left(\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \cdots\right)$$

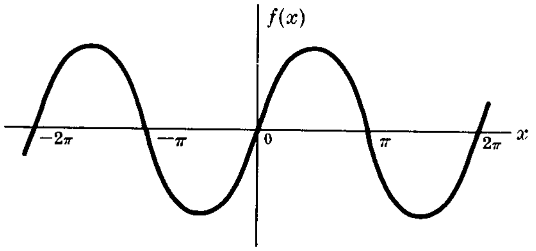


Fig. 23-12

## MISCELLANEOUS FOURIER SERIES

**23.19** $f(x) = \sin \mu x, \quad -\pi < x < \pi, \quad \mu \neq \text{integer}$

$$\frac{2 \sin \mu\pi}{\pi}\left(\frac{\sin x}{1^2 - \mu^2} - \frac{2 \sin 2x}{2^2 - \mu^2} + \frac{3 \sin 3x}{3^2 - \mu^2} - \cdots\right)$$

**23.20** $f(x) = \cos \mu x, \quad -\pi < x < \pi, \quad \mu \neq \text{integer}$

$$\frac{2\mu \sin \mu\pi}{\pi}\left(\frac{1}{2\mu^2} + \frac{\cos x}{1^2 - \mu^2} - \frac{\cos 2x}{2^2 - \mu^2} + \frac{\cos 3x}{3^2 - \mu^2} - \cdots\right)$$

**23.21** $f(x) = \tan^{-1}[(a \sin x)/(1 - a \cos x)], \quad -\pi < x < \pi, \quad |a| < 1$

$$a \sin x + \frac{a^2}{2} \sin 2x + \frac{a^3}{3} \sin 3x + \cdots$$

**23.22** $f(x) = \ln(1 - 2a \cos x + a^2), \quad -\pi < x < \pi, \quad |a| < 1$

$$-2\left(a \cos x + \frac{a^2}{2} \cos 2x + \frac{a^3}{3} \cos 3x + \cdots\right)$$

**23.23** $f(x) = \frac{1}{2} \tan^{-1}[(2a \sin x)/(1 - a^2)], \quad -\pi < x < \pi, \quad |a| < 1$

$$a \sin x + \frac{a^3}{3} \sin 3x + \frac{a^5}{5} \sin 5x + \cdots$$

**23.24** $f(x) = \frac{1}{2}\tan^{-1}[(2a\cos x)/(1-a^2)], \quad -\pi < x < \pi, \quad |a| < 1$

$$a\cos x - \frac{a^3}{3}\cos 3x + \frac{a^5}{5}\cos 5x - \cdots$$

**23.25** $f(x) = e^{\mu x}, \quad -\pi < x < \pi$

$$\frac{2\sinh\mu\pi}{\pi}\left(\frac{1}{2\mu} + \sum_{n=1}^{\infty}\frac{(-1)^n(\mu\cos nx - n\sin nx)}{\mu^2+n^2}\right)$$

**23.26** $f(x) = \sinh\mu x, \quad -\pi < x < \pi$

$$\frac{2\sinh\mu\pi}{\pi}\left(\frac{\sin x}{1^2+\mu^2} - \frac{2\sin 2x}{2^2+\mu^2} + \frac{3\sin 3x}{3^2+\mu^2} - \cdots\right)$$

**23.27** $f(x) = \cosh\mu x, \quad -\pi < x < \pi$

$$\frac{2\mu\sinh\mu\pi}{\pi}\left(\frac{1}{2\mu^2} - \frac{\cos x}{1^2+\mu^2} + \frac{\cos 2x}{2^2+\mu^2} - \frac{\cos 3x}{3^2+\mu^2} + \cdots\right)$$

**23.28** $f(x) = \ln|\sin\frac{1}{2}x|, \quad 0 < x < \pi$

$$-\left(\ln 2 + \frac{\cos x}{1} + \frac{\cos 2x}{2} + \frac{\cos 3x}{3} + \cdots\right)$$

**23.29** $f(x) = \ln|\cos\frac{1}{2}x|, \quad -\pi < x < \pi$

$$-\left(\ln 2 - \frac{\cos x}{1} + \frac{\cos 2x}{2} - \frac{\cos 3x}{3} + \cdots\right)$$

**23.30** $f(x) = \frac{1}{6}\pi^2 - \frac{1}{2}\pi x + \frac{1}{4}x^2, \quad 0 \leqq x \leqq 2\pi$

$$\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots$$

**23.31** $f(x) = \frac{1}{12}x(x-\pi)(x-2\pi), \quad 0 \leqq x \leqq 2\pi$

$$\frac{\sin x}{1^3} + \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} + \cdots$$

**23.32** $f(x) = \frac{1}{90}\pi^4 - \frac{1}{12}\pi^2x^2 + \frac{1}{12}\pi x^3 - \frac{1}{48}x^4, \quad 0 \leqq x \leqq 2\pi$

$$\frac{\cos x}{1^4} + \frac{\cos 2x}{2^4} + \frac{\cos 3x}{3^4} + \cdots$$

# 24 BESSEL FUNCTIONS

## BESSEL'S DIFFERENTIAL EQUATION

**24.1**
$$x^2y'' + xy' + (x^2 - n^2)y = 0 \qquad n \geqq 0$$

Solutions of this equation are called *Bessel functions of order n.*

## BESSEL FUNCTIONS OF THE FIRST KIND OF ORDER $n$

**24.2**
$$\begin{aligned} J_n(x) &= \frac{x^n}{2^n \Gamma(n+1)}\left\{1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2 \cdot 4(2n+2)(2n+4)} - \cdots\right\} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k!\,\Gamma(n+k+1)} \end{aligned}$$

**24.3**
$$\begin{aligned} J_{-n}(x) &= \frac{x^{-n}}{2^{-n} \Gamma(1-n)}\left\{1 - \frac{x^2}{2(2-2n)} + \frac{x^4}{2 \cdot 4(2-2n)(4-2n)} - \cdots\right\} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k-n}}{k!\,\Gamma(k+1-n)} \end{aligned}$$

**24.4**
$$J_{-n}(x) = (-1)^n J_n(x) \qquad n = 0, 1, 2, \ldots$$

If $n \neq 0, 1, 2, \ldots$, $J_n(x)$ and $J_{-n}(x)$ are linearly independent.

If $n \neq 0, 1, 2, \ldots$, $J_n(x)$ is bounded at $x = 0$ while $J_{-n}(x)$ is unbounded.

For $n = 0, 1$ we have

**24.5**
$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

**24.6**
$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} - \frac{x^7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \cdots$$

**24.7**
$$J_0'(x) = -J_1(x)$$

## BESSEL FUNCTIONS OF THE SECOND KIND OF ORDER $n$

**24.8**
$$Y_n(x) = \begin{cases} \dfrac{J_n(x) \cos n\pi - J_{-n}(x)}{\sin n\pi} & n \neq 0, 1, 2, \ldots \\[2ex] \lim\limits_{p \to n} \dfrac{J_p(x) \cos p\pi - J_{-p}(x)}{\sin p\pi} & n = 0, 1, 2, \ldots \end{cases}$$

This is also called *Weber's function* or *Neumann's function* [also denoted by $N_n(x)$].

For $n = 0, 1, 2, \ldots$, L'Hospital's rule yields

**24.9**
$$Y_n(x) = \frac{2}{\pi}\{\ln(x/2) + \gamma\} J_n(x) - \frac{1}{\pi}\sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}(x/2)^{2k-n} - \frac{1}{\pi}\sum_{k=0}^{\infty} (-1)^k\{\Phi(k) + \Phi(n+k)\}\frac{(x/2)^{2k+n}}{k!\,(n+k)!}$$

where $\gamma = .5772156\ldots$ is Euler's constant [page 1] and

**24.10**
$$\Phi(p) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p}, \qquad \Phi(0) = 0$$

For $n = 0$,

**24.11**
$$Y_0(x) = \frac{2}{\pi}\{\ln(x/2) + \gamma\} J_0(x) + \frac{2}{\pi}\left\{\frac{x^2}{2^2} - \frac{x^4}{2^2 4^2}(1 + \tfrac{1}{2}) + \frac{x^6}{2^2 4^2 6^2}(1 + \tfrac{1}{2} + \tfrac{1}{3}) - \cdots\right\}$$

**24.12**
$$Y_{-n}(x) = (-1)^n Y_n(x) \qquad n = 0, 1, 2, \ldots$$

For any value $n \geqq 0$, $J_n(x)$ is bounded at $x = 0$ while $Y_n(x)$ is unbounded.

## GENERAL SOLUTION OF BESSEL'S DIFFERENTIAL EQUATION

**24.13**
$$y = A J_n(x) + B J_{-n}(x) \qquad n \neq 0, 1, 2, \ldots$$

**24.14**
$$y = A J_n(x) + B Y_n(x) \qquad \text{all } n$$

**24.15**
$$y = A J_n(x) + B J_n(x) \int \frac{dx}{x J_n^2(x)} \qquad \text{all } n$$

where $A$ and $B$ are arbitrary constants.

## GENERATING FUNCTION FOR $J_n(x)$

**24.16**
$$e^{x(t-1/t)/2} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

## RECURRENCE FORMULAS FOR BESSEL FUNCTIONS

**24.17**
$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

**24.18**
$$J_n'(x) = \tfrac{1}{2}\{J_{n-1}(x) - J_{n+1}(x)\}$$

**24.19**
$$x J_n'(x) = x J_{n-1}(x) - n J_n(x)$$

**24.20**
$$x J_n'(x) = n J_n(x) - x J_{n+1}(x)$$

**24.21**
$$\frac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)$$

**24.22**
$$\frac{d}{dx}\{x^{-n} J_n(x)\} = -x^{-n} J_{n+1}(x)$$

The functions $Y_n(x)$ satisfy identical relations.

## BESSEL FUNCTIONS OF ORDER EQUAL TO HALF AN ODD INTEGER

In this case the functions are expressible in terms of sines and cosines.

**24.23** $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

**24.24** $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

**24.25** $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left( \frac{\sin x}{x} - \cos x \right)$

**24.26** $J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}} \left( \frac{\cos x}{x} + \sin x \right)$

**24.27** $J_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left\{ \left( \frac{3}{x^2} - 1 \right) \sin x - \frac{3}{x} \cos x \right\}$

**24.28** $J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}} \left\{ \frac{3}{x} \sin x + \left( \frac{3}{x^2} - 1 \right) \cos x \right\}$

For further results use the recurrence formula. Results for $Y_{1/2}(x), Y_{3/2}(x), \ldots$ are obtained from 24.8.

## HANKEL FUNCTIONS OF FIRST AND SECOND KINDS OF ORDER $n$

**24.29** $H_n^{(1)}(x) = J_n(x) + i\,Y_n(x)$

**24.30** $H_n^{(2)}(x) = J_n(x) - i\,Y_n(x)$

## BESSEL'S MODIFIED DIFFERENTIAL EQUATION

**24.31**
$$x^2 y'' + xy' - (x^2 + n^2)y = 0 \qquad n \geqq 0$$

Solutions of this equation are called *modified Bessel functions of order n.*

## MODIFIED BESSEL FUNCTIONS OF THE FIRST KIND OF ORDER $n$

**24.32**
$$I_n(x) = i^{-n} J_n(ix) = e^{-n\pi i/2} J_n(ix)$$
$$= \frac{x^n}{2^n \Gamma(n+1)} \left\{ 1 + \frac{x^2}{2(2n+2)} + \frac{x^4}{2 \cdot 4(2n+2)(2n+4)} + \cdots \right\} = \sum_{k=0}^{\infty} \frac{(x/2)^{n+2k}}{k!\, \Gamma(n+k+1)}$$

**24.33**
$$I_{-n}(x) = i^n J_{-n}(ix) = e^{n\pi i/2} J_{-n}(ix)$$
$$= \frac{x^{-n}}{2^{-n} \Gamma(1-n)} \left\{ 1 + \frac{x^2}{2(2-2n)} + \frac{x^4}{2 \cdot 4(2-2n)(4-2n)} + \cdots \right\} = \sum_{k=0}^{\infty} \frac{(x/2)^{2k-n}}{k!\, \Gamma(k+1-n)}$$

**24.34**
$$I_{-n}(x) = I_n(x) \qquad n = 0, 1, 2, \ldots$$

If $n \neq 0, 1, 2, \ldots$, then $I_n(x)$ and $I_{-n}(x)$ are linearly independent.

For $n = 0, 1$, we have

**24.35** $I_0(x) = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$

**24.36** $I_1(x) = \frac{x}{2} + \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} + \frac{x^7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \cdots$

**24.37** $I_0'(x) = I_1(x)$

## MODIFIED BESSEL FUNCTIONS OF THE SECOND KIND OF ORDER $n$

**24.38**
$$K_n(x) = \begin{cases} \dfrac{\pi}{2\sin n\pi}\{I_{-n}(x) - I_n(x)\} & n \neq 0, 1, 2, \ldots \\[2ex] \lim\limits_{p\to n} \dfrac{\pi}{2\sin p\pi}\{I_{-p}(x) - I_p(x)\} & n = 0, 1, 2, \ldots \end{cases}$$

For $n = 0, 1, 2, \ldots,$ L'Hospital's rule yields

**24.39**
$$K_n(x) = (-1)^{n+1}\{\ln(x/2) + \gamma\}I_n(x) + \frac{1}{2}\sum_{k=0}^{n-1}(-1)^k(n-k-1)!\,(x/2)^{2k-n}$$
$$+ \frac{(-1)^n}{2}\sum_{k=0}^{\infty}\frac{(x/2)^{n+2k}}{k!\,(n+k)!}\{\Phi(k) + \Phi(n+k)\}$$

where $\Phi(p)$ is given by 24.10.

For $n = 0$,

**24.40**
$$K_0(x) = -\{\ln(x/2) + \gamma\}I_0(x) + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2}(1 + \tfrac{1}{2}) + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2}(1 + \tfrac{1}{2} + \tfrac{1}{3}) + \cdots$$

**24.41**
$$K_{-n}(x) = K_n(x) \qquad n = 0, 1, 2, \ldots$$

## GENERAL SOLUTION OF BESSEL'S MODIFIED EQUATION

**24.42**
$$y = AI_n(x) + BI_{-n}(x) \qquad n \neq 0, 1, 2, \ldots$$

**24.43**
$$y = AI_n(x) + BK_n(x) \qquad \text{all } n$$

**24.44**
$$y = AI_n(x) + BI_n(x)\int\frac{dx}{xI_n^2(x)} \qquad \text{all } n$$

where $A$ and $B$ are arbitrary constants.

## GENERATING FUNCTION FOR $I_n(x)$

**24.45**
$$e^{x(t+1/t)/2} = \sum_{n=-\infty}^{\infty} I_n(x)t^n$$

## RECURRENCE FORMULAS FOR MODIFIED BESSEL FUNCTIONS

**24.46** $I_{n+1}(x) = I_{n-1}(x) - \dfrac{2n}{x}I_n(x)$

**24.47** $I_n'(x) = \tfrac{1}{2}\{I_{n-1}(x) + I_{n+1}(x)\}$

**24.48** $xI_n'(x) = xI_{n-1}(x) - nI_n(x)$

**24.49** $xI_n'(x) = xI_{n+1}(x) + nI_n(x)$

**24.50** $\dfrac{d}{dx}\{x^nI_n(x)\} = x^nI_{n-1}(x)$

**24.51** $\dfrac{d}{dx}\{x^{-n}I_n(x)\} = x^{-n}I_{n+1}(x)$

**24.52** $K_{n+1}(x) = K_{n-1}(x) + \dfrac{2n}{x}K_n(x)$

**24.53** $K_n'(x) = \tfrac{1}{2}\{K_{n-1}(x) + K_{n+1}(x)\}$

**24.54** $xK_n'(x) = -xK_{n-1}(x) - nK_n(x)$

**24.55** $xK_n'(x) = nK_n(x) - xK_{n+1}(x)$

**24.56** $\dfrac{d}{dx}\{x^nK_n(x)\} = -x^nK_{n-1}(x)$

**24.57** $\dfrac{d}{dx}\{x^{-n}K_n(x)\} = -x^{-n}K_{n+1}(x)$

## MODIFIED BESSEL FUNCTIONS OF ORDER EQUAL TO HALF AN ODD INTEGER

In this case the functions are expressible in terms of hyperbolic sines and cosines.

**24.58** $$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sinh x$$

**24.59** $$I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cosh x$$

**24.60** $$I_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\cosh x - \frac{\sinh x}{x}\right)$$

**24.61** $$I_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\sinh x - \frac{\cosh x}{x}\right)$$

**24.62** $$I_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left\{\left(\frac{3}{x^2}+1\right)\sinh x - \frac{3}{x}\cosh x\right\}$$

**24.63** $$I_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left\{\left(\frac{3}{x^2}+1\right)\cosh x - \frac{3}{x}\sinh x\right\}$$

For further results use the recurrence formula 24.46. Results for $K_{1/2}(x), K_{3/2}(x), \ldots$ are obtained from 24.38.

## Ber AND Bei FUNCTIONS

The real and imaginary parts of $J_n(xe^{3\pi i/4})$ are denoted by $\operatorname{Ber}_n(x)$ and $\operatorname{Bei}_n(x)$ where

**24.64** $$\operatorname{Ber}_n(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+n}}{k!\,\Gamma(n+k+1)} \cos\frac{(3n+2k)\pi}{4}$$

**24.65** $$\operatorname{Bei}_n(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+n}}{k!\,\Gamma(n+k+1)} \sin\frac{(3n+2k)\pi}{4}$$

If $n = 0$,

**24.66** $$\operatorname{Ber}(x) = 1 - \frac{(x/2)^4}{2!^2} + \frac{(x/2)^8}{4!^2} - \cdots$$

**24.67** $$\operatorname{Bei}(x) = (x/2)^2 - \frac{(x/2)^6}{3!^2} + \frac{(x/2)^{10}}{5!^2} - \cdots$$

## Ker AND Kei FUNCTIONS

The real and imaginary parts of $e^{-n\pi i/2}K_n(xe^{\pi i/4})$ are denoted by $\operatorname{Ker}_n(x)$ and $\operatorname{Kei}_n(x)$ where

**24.68** $$\begin{aligned}\operatorname{Ker}_n(x) = {} & -\{\ln(x/2)+\gamma\}\operatorname{Ber}_n(x) + \tfrac{1}{4}\pi \operatorname{Bei}_n(x) \\ & + \frac{1}{2}\sum_{k=0}^{n-1} \frac{(n-k-1)!\,(x/2)^{2k-n}}{k!} \cos\frac{(3n+2k)\pi}{4} \\ & + \frac{1}{2}\sum_{k=0}^{\infty} \frac{(x/2)^{n+2k}}{k!\,(n+k)!}\{\Phi(k)+\Phi(n+k)\} \cos\frac{(3n+2k)\pi}{4}\end{aligned}$$

**24.69** $$\begin{aligned}\operatorname{Kei}_n(x) = {} & -\{\ln(x/2)+\gamma\}\operatorname{Bei}_n(x) - \tfrac{1}{4}\pi \operatorname{Ber}_n(x) \\ & - \frac{1}{2}\sum_{k=0}^{n-1} \frac{(n-k-1)!\,(x/2)^{2k-n}}{k!} \sin\frac{(3n+2k)\pi}{4} \\ & + \frac{1}{2}\sum_{k=0}^{\infty} \frac{(x/2)^{n+2k}}{k!\,(n+k)!}\{\Phi(k)+\Phi(n+k)\} \sin\frac{(3n+2k)\pi}{4}\end{aligned}$$

and $\Phi$ is given by 24.10, page 137.

If $n = 0$,

**24.70** $$\operatorname{Ker}(x) = -\{\ln(x/2)+\gamma\}\operatorname{Ber}(x) + \frac{\pi}{4}\operatorname{Bei}(x) + 1 - \frac{(x/2)^4}{2!^2}(1+\tfrac{1}{2}) + \frac{(x/2)^8}{4!^2}(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}) - \cdots$$

**24.71** $$\operatorname{Kei}(x) = -\{\ln(x/2)+\gamma\}\operatorname{Bei}(x) - \frac{\pi}{4}\operatorname{Ber}(x) + (x/2)^2 - \frac{(x/2)^6}{3!^2}(1+\tfrac{1}{2}+\tfrac{1}{3}) + \cdots$$

## DIFFERENTIAL EQUATION FOR Ber, Bei, Ker, Kei FUNCTIONS

**24.72**
$$x^2y'' + xy' - (ix^2 + n^2)y = 0$$

The general solution of this equation is

**24.73**
$$y = A\{\mathrm{Ber}_n(x) + i\,\mathrm{Bei}_n(x)\} + B\{\mathrm{Ker}_n(x) + i\,\mathrm{Kei}_n(x)\}$$

## GRAPHS OF BESSEL FUNCTIONS

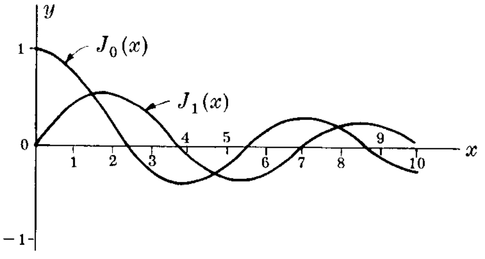


Fig. 24-1

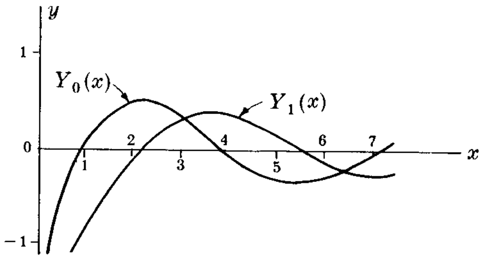


Fig. 24-2

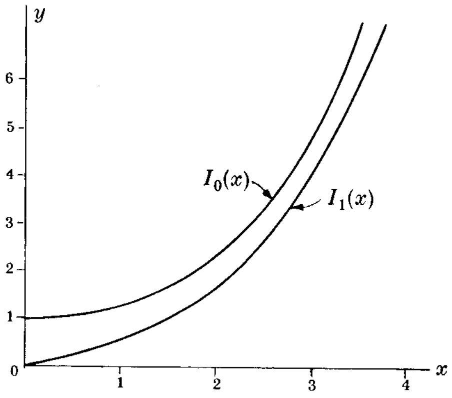


Fig. 24-3

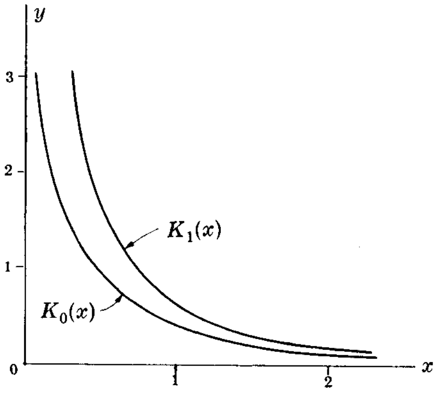


Fig. 24-4

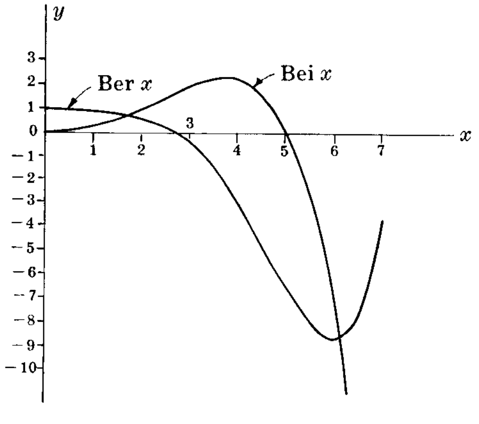


Fig. 24-5

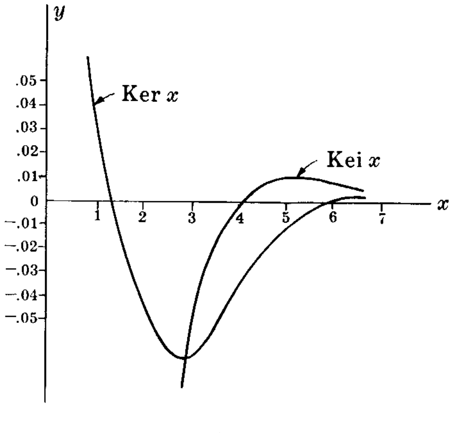


Fig. 24-6

## INDEFINITE INTEGRALS INVOLVING BESSEL FUNCTIONS

**24.74** $$\int x\,J_0(x)\,dx = x\,J_1(x)$$

**24.75** $$\int x^2 J_0(x)\,dx = x^2 J_1(x) + x\,J_0(x) - \int J_0(x)\,dx$$

**24.76** $$\int x^m J_0(x)\,dx = x^m J_1(x) + (m-1)x^{m-1} J_0(x) - (m-1)^2 \int x^{m-2} J_0(x)\,dx$$

**24.77** $$\int \frac{J_0(x)}{x^2}\,dx = J_1(x) - \frac{J_0(x)}{x} - \int J_0(x)\,dx$$

**24.78** $$\int \frac{J_0(x)}{x^m}\,dx = \frac{J_1(x)}{(m-1)^2 x^{m-2}} - \frac{J_0(x)}{(m-1)x^{m-1}} - \frac{1}{(m-1)^2}\int \frac{J_0(x)}{x^{m-2}}\,dx$$

**24.79** $$\int J_1(x)\,dx = -J_0(x)$$

**24.80** $$\int x\,J_1(x)\,dx = -x\,J_0(x) + \int J_0(x)\,dx$$

**24.81** $$\int x^m J_1(x)\,dx = -x^m J_0(x) + m \int x^{m-1} J_0(x)\,dx$$

**24.82** $$\int \frac{J_1(x)}{x}\,dx = -J_1(x) + \int J_0(x)\,dx$$

**24.83** $$\int \frac{J_1(x)}{x^m}\,dx = -\frac{J_1(x)}{m x^{m-1}} + \frac{1}{m}\int \frac{J_0(x)}{x^{m-1}}\,dx$$

**24.84** $$\int x^n J_{n-1}(x)\,dx = x^n J_n(x)$$

**24.85** $$\int x^{-n} J_{n+1}(x)\,dx = -x^{-n} J_n(x)$$

**24.86** $$\int x^m J_n(x)\,dx = -x^m J_{n-1}(x) + (m+n-1)\int x^{m-1} J_{n-1}(x)\,dx$$

**24.87** $$\int x\,J_n(\alpha x) J_n(\beta x)\,dx = \frac{x\{\alpha\,J_n(\beta x)\,J_n'(\alpha x) - \beta\,J_n(\alpha x)\,J_n'(\beta x)\}}{\beta^2 - \alpha^2}$$

**24.88** $$\int x\,J_n^2(\alpha x)\,dx = \frac{x^2}{2}\{J_n'(\alpha x)\}^2 + \frac{x^2}{2}\left(1 - \frac{n^2}{\alpha^2 x^2}\right)\{J_n(\alpha x)\}^2$$

The above results also hold if we replace $J_n(x)$ by $Y_n(x)$ or, more generally, $A\,J_n(x) + B\,Y_n(x)$ where $A$ and $B$ are constants.

## DEFINITE INTEGRALS INVOLVING BESSEL FUNCTIONS

**24.89** $$\int_0^\infty e^{-ax} J_0(bx)\,dx = \frac{1}{\sqrt{a^2+b^2}}$$

**24.90** $$\int_0^\infty e^{-ax} J_n(bx)\,dx = \frac{(\sqrt{a^2+b^2}-a)^n}{b^n\sqrt{a^2+b^2}} \qquad n > -1$$

**24.91** $$\int_0^\infty \cos ax\,J_0(bx)\,dx = \begin{cases} \dfrac{1}{\sqrt{a^2-b^2}} & a > b \\ 0 & a < b \end{cases}$$

**24.92** $\displaystyle\int_0^\infty J_n(bx)\,dx \;=\; \frac{1}{b} \qquad n > -1$

**24.93** $\displaystyle\int_0^\infty \frac{J_n(bx)}{x}\,dx \;=\; \frac{1}{n} \qquad n = 1, 2, 3, \ldots$

**24.94** $\displaystyle\int_0^\infty e^{-ax} J_0(b\sqrt{x})\,dx \;=\; \frac{e^{-b^2/4a}}{a}$

**24.95** $\displaystyle\int_0^1 x\,J_n(\alpha x)\,J_n(\beta x)\,dx \;=\; \frac{\alpha\,J_n(\beta)\,J_n'(\alpha) - \beta\,J_n(\alpha)\,J_n'(\beta)}{\beta^2 - \alpha^2}$

**24.96** $\displaystyle\int_0^1 x\,J_n^2(\alpha x)\,dx \;=\; \tfrac{1}{2}\{J_n'(\alpha)\}^2 + \tfrac{1}{2}(1 - n^2/\alpha^2)\{J_n(\alpha)\}^2$

**24.97** $\displaystyle\int_0^1 x\,J_0(\alpha x)\,I_0(\beta x)\,dx \;=\; \frac{\beta\,J_0(\alpha)\,I_0'(\beta) - \alpha\,J_0'(\alpha)\,I_0(\beta)}{\alpha^2 + \beta^2}$

## INTEGRAL REPRESENTATIONS FOR BESSEL FUNCTIONS

**24.98** $\displaystyle J_0(x) \;=\; \frac{1}{\pi}\int_0^\pi \cos(x \sin\theta)\,d\theta$

**24.99** $\displaystyle J_n(x) \;=\; \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta, \qquad n = \text{integer}$

**24.100** $\displaystyle J_n(x) \;=\; \frac{x^n}{2^n\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\int_0^\pi \cos(x\sin\theta)\cos^{2n}\theta\,d\theta, \qquad n > -\tfrac{1}{2}$

**24.101** $\displaystyle Y_0(x) \;=\; -\frac{2}{\pi}\int_0^\infty \cos(x\cosh u)\,du$

**24.102** $\displaystyle I_0(x) \;=\; \frac{1}{\pi}\int_0^\pi \cosh(x\sin\theta)\,d\theta \;=\; \frac{1}{2\pi}\int_0^{2\pi} e^{x\sin\theta}\,d\theta$

## ASYMPTOTIC EXPANSIONS

**24.103** $\displaystyle J_n(x) \;\sim\; \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$ where $x$ is large

**24.104** $\displaystyle Y_n(x) \;\sim\; \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$ where $x$ is large

**24.105** $\displaystyle J_n(x) \;\sim\; \frac{1}{\sqrt{2\pi n}}\left(\frac{ex}{2n}\right)^n$ where $n$ is large

**24.106** $\displaystyle Y_n(x) \;\sim\; -\sqrt{\frac{2}{\pi n}}\left(\frac{ex}{2n}\right)^{-n}$ where $n$ is large

**24.107** $\displaystyle I_n(x) \;\sim\; \frac{e^x}{\sqrt{2\pi x}}$ where $x$ is large

**24.108** $\displaystyle K_n(x) \;\sim\; \frac{e^{-x}}{\sqrt{2\pi x}}$ where $x$ is large

## ORTHOGONAL SERIES OF BESSEL FUNCTIONS

Let $\lambda_1, \lambda_2, \lambda_3, \ldots$ be the positive roots of $R\,J_n(x) + Sx\,J_n'(x) = 0$, $n > -1$. Then the following series expansions hold under the conditions indicated.

$S = 0$, $R \neq 0$, i.e. $\lambda_1, \lambda_2, \lambda_3, \ldots$ are positive roots of $J_n(x) = 0$

**24.109**
$$f(x) = A_1 J_n(\lambda_1 x) + A_2 J_n(\lambda_2 x) + A_3 J_n(\lambda_3 x) + \cdots$$

where

**24.110**
$$A_k = \frac{2}{J_{n+1}^2(\lambda_k)} \int_0^1 x\, f(x)\, J_n(\lambda_k x)\, dx$$

In particular if $n = 0$,

**24.111**
$$f(x) = A_1 J_0(\lambda_1 x) + A_2 J_0(\lambda_2 x) + A_3 J_0(\lambda_3 x) + \cdots$$

where

**24.112**
$$A_k = \frac{2}{J_1^2(\lambda_k)} \int_0^1 x\, f(x)\, J_0(\lambda_k x)\, dx$$

$R/S > -n$

**24.113**
$$f(x) = A_1 J_n(\lambda_1 x) + A_2 J_n(\lambda_2 x) + A_3 J_n(\lambda_3 x) + \cdots$$

where

**24.114**
$$A_k = \frac{2}{J_n^2(\lambda_k) - J_{n-1}(\lambda_k)\, J_{n+1}(\lambda_k)} \int_0^1 x\, f(x)\, J_n(\lambda_k x)\, dx$$

In particular if $n = 0$,

**24.115**
$$f(x) = A_1 J_0(\lambda_1 x) + A_2 J_0(\lambda_2 x) + A_3 J_0(\lambda_3 x) + \cdots$$

where

**24.116**
$$A_k = \frac{2}{J_0^2(\lambda_k) + J_1^2(\lambda_k)} \int_0^1 x\, f(x)\, J_0(\lambda_k x)\, dx$$

$R/S = -n$

**24.117**
$$f(x) = A_0 x^n + A_1 J_n(\lambda_1 x) + A_2 J_n(\lambda_2 x) + \cdots$$

where

**24.118**
$$\begin{cases} A_0 = 2(n+1) \displaystyle\int_0^1 x^{n+1} f(x)\, dx \\ A_k = \dfrac{2}{J_n^2(\lambda_k) - J_{n-1}(\lambda_k)\, J_{n+1}(\lambda_k)} \displaystyle\int_0^1 x\, f(x)\, J_n(\lambda_k x)\, dx \end{cases}$$

In particular if $n = 0$ so that $R = 0$ [i.e. $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive roots of $J_1(x) = 0$],

**24.119**
$$f(x) = A_0 + A_1 J_0(\lambda_1 x) + A_2 J_0(\lambda_2 x) + \cdots$$

where

**24.120**
$$\begin{cases} A_0 = 2 \displaystyle\int_0^1 x\, f(x)\, dx \\ A_k = \dfrac{2}{J_0^2(\lambda_k)} \displaystyle\int_0^1 x\, f(x)\, J_0(\lambda_k x)\, dx \end{cases}$$

$R/S < -n$

In this case there are two pure imaginary roots $\pm i\lambda_0$ as well as the positive roots $\lambda_1, \lambda_2, \lambda_3, \ldots$ and we have

**24.121** $$f(x) = A_0 I_n(\lambda_0 x) + A_1 J_n(\lambda_1 x) + A_2 J_n(\lambda_2 x) + \cdots$$

where

**24.122** $$\begin{cases} A_0 = \dfrac{2}{I_n^2(\lambda_0) + I_{n-1}(\lambda_0)\, I_{n+1}(\lambda_0)} \displaystyle\int_0^1 x\, f(x)\, I_n(\lambda_0 x)\, dx \\ A_k = \dfrac{2}{J_n^2(\lambda_k) - J_{n-1}(\lambda_k)\, J_{n+1}(\lambda_k)} \displaystyle\int_0^1 x\, f(x)\, J_n(\lambda_k x)\, dx \end{cases}$$

## MISCELLANEOUS RESULTS

**24.123** $\cos(x \sin\theta) = J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \cdots$

**24.124** $\sin(x \sin\theta) = 2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + 2J_5(x)\sin 5\theta + \cdots$

**24.125** $J_n(x+y) = \sum_{k=-\infty}^{\infty} J_k(x)\, J_{n-k}(y) \qquad n = 0, \pm 1, \pm 2, \ldots$

This is called the *addition formula* for Bessel functions.

**24.126** $1 = J_0(x) + 2J_2(x) + \cdots + 2J_{2n}(x) + \cdots$

**24.127** $x = 2\{J_1(x) + 3J_3(x) + 5J_5(x) + \cdots + (2n+1)J_{2n+1}(x) + \cdots\}$

**24.128** $x^2 = 2\{4J_2(x) + 16J_4(x) + 36J_6(x) + \cdots + (2n)^2 J_{2n}(x) + \cdots\}$

**24.129** $\dfrac{x J_1(x)}{4} = J_2(x) - 2J_4(x) + 3J_6(x) - \cdots$

**24.130** $1 = J_0^2(x) + 2J_1^2(x) + 2J_2^2(x) + 2J_3^2(x) + \cdots$

**24.131** $J_n''(x) = \frac{1}{4}\{J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)\}$

**24.132** $J_n'''(x) = \frac{1}{8}\{J_{n-3}(x) - 3J_{n-1}(x) + 3J_{n+1}(x) - J_{n+3}(x)\}$

Formulas 24.131 and 24.132 can be generalized.

**24.133** $J_n'(x)\, J_{-n}(x) - J_{-n}'\, J_n(x) = \dfrac{2\sin n\pi}{\pi x}$

**24.134** $J_n(x)\, J_{-n+1}(x) + J_{-n}(x)\, J_{n-1}(x) = \dfrac{2\sin n\pi}{\pi x}$

**24.135** $J_{n+1}(x)\, Y_n(x) - J_n(x)\, Y_{n+1}(x) = J_n(x)\, Y_n'(x) - J_n'(x)\, Y_n(x) = \dfrac{2}{\pi x}$

**24.136** $\sin x = 2\{J_1(x) - J_3(x) + J_5(x) - \cdots\}$

**24.137** $\cos x = J_0(x) - 2J_2(x) + 2J_4(x) - \cdots$

**24.138** $\sinh x = 2\{I_1(x) + I_3(x) + I_5(x) + \cdots\}$

**24.139** $\cosh x = I_0(x) + 2\{I_2(x) + I_4(x) + I_6(x) + \cdots\}$

# 25 LEGENDRE FUNCTIONS

## LEGENDRE'S DIFFERENTIAL EQUATION

**25.1** $$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

Solutions of this equation are called *Legendre functions of order n.*

## LEGENDRE POLYNOMIALS

If $n = 0, 1, 2, \ldots,$ solutions of 25.1 are Legendre polynomials $P_n(x)$ given by *Rodrigue's formula*

**25.2** $$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

## SPECIAL LEGENDRE POLYNOMIALS

**25.3** $P_0(x) = 1$

**25.4** $P_1(x) = x$

**25.5** $P_2(x) = \frac{1}{2}(3x^2-1)$

**25.6** $P_3(x) = \frac{1}{2}(5x^3-3x)$

**25.7** $P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$

**25.8** $P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$

**25.9** $P_6(x) = \frac{1}{16}(231x^6-315x^4+105x^2-5)$

**25.10** $P_7(x) = \frac{1}{16}(429x^7-693x^5+315x^3-35x)$

## LEGENDRE POLYNOMIALS IN TERMS OF $\theta$ WHERE $x = \cos\theta$

**25.11** $P_0(\cos\theta) = 1$

**25.12** $P_1(\cos\theta) = \cos\theta$

**25.13** $P_2(\cos\theta) = \frac{1}{4}(1 + 3\cos 2\theta)$

**25.14** $P_3(\cos\theta) = \frac{1}{8}(3\cos\theta + 5\cos 3\theta)$

**25.15** $P_4(\cos\theta) = \frac{1}{64}(9 + 20\cos 2\theta + 35\cos 4\theta)$

**25.16** $P_5(\cos\theta) = \frac{1}{128}(30\cos\theta + 35\cos 3\theta + 63\cos 5\theta)$

**25.17** $P_6(\cos\theta) = \frac{1}{512}(50 + 105\cos 2\theta + 126\cos 4\theta + 231\cos 6\theta)$

**25.18** $P_7(\cos\theta) = \frac{1}{1024}(175\cos\theta + 189\cos 3\theta + 231\cos 5\theta + 429\cos 7\theta)$

## GENERATING FUNCTION FOR LEGENDRE POLYNOMIALS

**25.19** $$\frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

## RECURRENCE FORMULAS FOR LEGENDRE POLYNOMIALS

**25.20** $$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$$

**25.21** $$P'_{n+1}(x) - xP'_n(x) = (n+1)P_n(x)$$

**25.22** $$xP'_n(x) - P'_{n-1}(x) = nP_n(x)$$

**25.23** $$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$$

**25.24** $$(x^2-1)P'_n(x) = nxP_n(x) - nP_{n-1}(x)$$

## ORTHOGONALITY OF LEGENDRE POLYNOMIALS

**25.25** $$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0 \qquad m \neq n$$

**25.26** $$\int_{-1}^{1} \{P_n(x)\}^2\,dx = \frac{2}{2n+1}$$

Because of 25.25, $P_m(x)$ and $P_n(x)$ are called *orthogonal* in $-1 \leqq x \leqq 1$.

## ORTHOGONAL SERIES OF LEGENDRE POLYNOMIALS

**25.27** $$f(x) = A_0P_0(x) + A_1P_1(x) + A_2P_2(x) + \cdots$$

where

**25.28** $$A_k = \frac{2k+1}{2}\int_{-1}^{1} f(x)P_k(x)\,dx$$

## SPECIAL RESULTS INVOLVING LEGENDRE POLYNOMIALS

**25.29** $P_n(1) = 1$ **25.30** $P_n(-1) = (-1)^n$ **25.31** $P_n(-x) = (-1)^nP_n(x)$

**25.32** $$P_n(0) = \begin{cases} 0 & n \text{ odd} \\ (-1)^{n/2}\dfrac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n} & n \text{ even} \end{cases}$$

**25.33** $$P_n(x) = \frac{1}{\pi}\int_0^{\pi} (x + \sqrt{x^2-1}\cos\phi)^n\,d\phi$$

**25.34** $$\int P_n(x)\,dx = \frac{P_{n+1}(x) - P_{n-1}(x)}{2n+1}$$

**25.35** $$|P_n(x)| \leqq 1$$

**25.36** $$P_n(x) = \frac{1}{2^{n+1}\pi i}\oint_C \frac{(z^2-1)^n}{(z-x)^{n+1}}\,dz$$

where $C$ is a simple closed curve having $x$ as interior point.

## GENERAL SOLUTION OF LEGENDRE'S EQUATION

The general solution of Legendre's equation is

**25.37** $$y = A\,U_n(x) + B\,V_n(x)$$

where

**25.38** $$U_n(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \cdots$$

**25.39** $$V_n(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \cdots$$

These series converge for $-1 < x < 1$.

## LEGENDRE FUNCTIONS OF THE SECOND KIND

If $n = 0, 1, 2, \ldots$ one of the series 25.38, 25.39 terminates. In such cases,

**25.40** $$P_n(x) = \begin{cases} U_n(x)/U_n(1) & n = 0, 2, 4, \ldots \\ V_n(x)/V_n(1) & n = 1, 3, 5, \ldots \end{cases}$$

where

**25.41** $$U_n(1) = (-1)^{n/2}\,2^n\left[\left(\frac{n}{2}\right)!\right]^2\Big/ n! \qquad n = 0, 2, 4, \ldots$$

**25.42** $$V_n(1) = (-1)^{(n-1)/2}\,2^{n-1}\left[\left(\frac{n-1}{2}\right)!\right]^2\Big/ n! \qquad n = 1, 3, 5, \ldots$$

The nonterminating series in such case with a suitable multiplicative constant is denoted by $Q_n(x)$ and is called *Legendre's function of the second kind of order n.* We define

**25.43** $$Q_n(x) = \begin{cases} U_n(1)\,V_n(x) & n = 0, 2, 4, \ldots \\ -V_n(1)\,U_n(x) & n = 1, 3, 5, \ldots \end{cases}$$

## SPECIAL LEGENDRE FUNCTIONS OF THE SECOND KIND

**25.44** $$Q_0(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

**25.45** $$Q_1(x) = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1$$

**25.46** $$Q_2(x) = \frac{3x^2-1}{4}\ln\left(\frac{1+x}{1-x}\right) - \frac{3x}{2}$$

**25.47** $$Q_3(x) = \frac{5x^3-3x}{4}\ln\left(\frac{1+x}{1-x}\right) - \frac{5x^2}{2} + \frac{2}{3}$$

The functions $Q_n(x)$ satisfy recurrence formulas exactly analogous to 25.20 through 25.24.

Using these, the general solution of Legendre's equation can also be written

**25.48** $$y = A\,P_n(x) + B\,Q_n(x)$$

# 26 ASSOCIATED LEGENDRE FUNCTIONS

## LEGENDRE'S ASSOCIATED DIFFERENTIAL EQUATION

**26.1**
$$(1-x^2)y'' - 2xy' + \left\{n(n+1) - \frac{m^2}{1-x^2}\right\}y = 0$$

Solutions of this equation are called *associated Legendre functions.* We restrict ourselves to the important case where $m, n$ are nonnegative integers.

## ASSOCIATED LEGENDRE FUNCTIONS OF THE FIRST KIND

**26.2**
$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) = \frac{(1-x^2)^{m/2}}{2^n n!}\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n$$

where $P_n(x)$ are Legendre polynomials [page 146]. We have

**26.3**
$$P_n^0(x) = P_n(x)$$

**26.4**
$$P_n^m(x) = 0 \qquad \text{if } m > n$$

## SPECIAL ASSOCIATED LEGENDRE FUNCTIONS OF THE FIRST KIND

**26.5** $P_1^1(x) = (1-x^2)^{1/2}$

**26.6** $P_2^1(x) = 3x(1-x^2)^{1/2}$

**26.7** $P_2^2(x) = 3(1-x^2)$

**26.8** $P_3^1(x) = \frac{3}{2}(5x^2-1)(1-x^2)^{1/2}$

**26.9** $P_3^2(x) = 15x(1-x^2)$

**26.10** $P_3^3(x) = 15(1-x^2)^{3/2}$

## GENERATING FUNCTION FOR $P_n^m(x)$

**26.11**
$$\frac{(2m)!\,(1-x^2)^{m/2}t^m}{2^m m!\,(1-2tx+t^2)^{m+1/2}} = \sum_{n=m}^{\infty} P_n^m(x)t^n$$

## RECURRENCE FORMULAS

**26.12**
$$(n+1-m)P_{n+1}^m(x) - (2n+1)xP_n^m(x) + (n+m)P_{n-1}^m(x) = 0$$

**26.13**
$$P_n^{m+2}(x) - \frac{2(m+1)x}{(1-x^2)^{1/2}}P_n^{m+1}(x) + (n-m)(n+m+1)P_n^m(x) = 0$$

## ORTHOGONALITY OF $P_n^m(x)$

**26.14** $$\int_{-1}^{1} P_n^m(x)\, P_l^m(x)\, dx \;=\; 0 \qquad \text{if } n \neq l$$

**26.15** $$\int_{-1}^{1} \{P_n^m(x)\}^2\, dx \;=\; \frac{2}{2n+1}\,\frac{(n+m)!}{(n-m)!}$$

## ORTHOGONAL SERIES

**26.16** $$f(x) \;=\; A_m P_m^m(x) + A_{m+1} P_{m+1}^m(x) + A_{m+2} P_{m+2}^m(x) + \cdots$$

where

**26.17** $$A_k \;=\; \frac{2k+1}{2}\,\frac{(k-m)!}{(k+m)!} \int_{-1}^{1} f(x)\, P_k^m(x)\, dx$$

## ASSOCIATED LEGENDRE FUNCTIONS OF THE SECOND KIND

**26.18** $$Q_n^m(x) \;=\; (1-x^2)^{m/2}\, \frac{d^m}{dx^m} Q_n(x)$$

where $Q_n(x)$ are Legendre functions of the second kind [page 148].

These functions are unbounded at $x = \pm 1$, whereas $P_n^m(x)$ are bounded at $x = \pm 1$.

The functions $Q_n^m(x)$ satisfy the same recurrence relations as $P_n^m(x)$ [see 26.12 and 26.13].

## GENERAL SOLUTION OF LEGENDRE'S ASSOCIATED EQUATION

**26.19** $$y \;=\; A\, P_n^m(x) + B\, Q_n^m(x)$$

# 27

# HERMITE POLYNOMIALS

## HERMITE'S DIFFERENTIAL EQUATION

**27.1**
$$y'' - 2xy' + 2ny = 0$$

## HERMITE POLYNOMIALS

If $n = 0, 1, 2, \ldots$ then solutions of Hermite's equation are Hermite polynomials $H_n(x)$ given by *Rodrigue's formula*

**27.2**
$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$$

## SPECIAL HERMITE POLYNOMIALS

**27.3** $H_0(x) = 1$

**27.4** $H_1(x) = 2x$

**27.5** $H_2(x) = 4x^2 - 2$

**27.6** $H_3(x) = 8x^3 - 12x$

**27.7** $H_4(x) = 16x^4 - 48x^2 + 12$

**27.8** $H_5(x) = 32x^5 - 160x^3 + 120x$

**27.9** $H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120$

**27.10** $H_7(x) = 128x^7 - 1344x^5 + 3360x^3 - 1680x$

## GENERATING FUNCTION

**27.11**
$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)\, t^n}{n!}$$

## RECURRENCE FORMULAS

**27.12**
$$H_{n+1}(x) = 2x\, H_n(x) - 2n\, H_{n-1}(x)$$

**27.13**
$$H_n'(x) = 2n\, H_{n-1}(x)$$

## ORTHOGONALITY OF HERMITE POLYNOMIALS

**27.14** $$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\,dx = 0 \qquad m \neq n$$

**27.15** $$\int_{-\infty}^{\infty} e^{-x^2} \{H_n(x)\}^2\,dx = 2^n n!\sqrt{\pi}$$

## ORTHOGONAL SERIES

**27.16** $$f(x) = A_0 H_0(x) + A_1 H_1(x) + A_2 H_2(x) + \cdots$$

where

**27.17** $$A_k = \frac{1}{2^k k!\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} f(x) H_k(x)\,dx$$

## SPECIAL RESULTS

**27.18** $$H_n(x) = (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} - \cdots$$

**27.19** $H_n(-x) = (-1)^n H_n(x)$ **27.20** $H_{2n-1}(0) = 0$

**27.21** $$H_{2n}(0) = (-1)^n 2^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)$$

**27.22** $$\int_0^x H_n(t)\,dt = \frac{H_{n+1}(x)}{2(n+1)} - \frac{H_{n+1}(0)}{2(n+1)}$$

**27.23** $$\frac{d}{dx}\{e^{-x^2} H_n(x)\} = -e^{-x^2} H_{n+1}(x)$$

**27.24** $$\int_0^x e^{-t^2} H_n(t)\,dt = H_{n-1}(0) - e^{-x^2} H_{n-1}(x)$$

**27.25** $$\int_{-\infty}^{\infty} t^n e^{-t^2} H_n(xt)\,dt = \sqrt{\pi}\,n!\,P_n(x)$$

**27.26** $$H_n(x+y) = \sum_{k=0}^{n} \frac{1}{2^{n/2}} \binom{n}{k} H_k(x\sqrt{2})\,H_{n-k}(y\sqrt{2})$$

This is called the *addition formula* for Hermite polynomials.

**27.27** $$\sum_{k=0}^{n} \frac{H_k(x) H_k(y)}{2^k k!} = \frac{H_{n+1}(x) H_n(y) - H_n(x) H_{n+1}(y)}{2^{n+1} n!\,(x-y)}$$

# 28 LAGUERRE POLYNOMIALS

## LAGUERRE'S DIFFERENTIAL EQUATION

**28.1** $$xy'' + (1-x)y' + ny = 0$$

## LAGUERRE POLYNOMIALS

If $n = 0, 1, 2, \ldots$ then solutions of Laguerre's equation are Laguerre polynomials $L_n(x)$ and are given by *Rodrigue's formula.*

**28.2** $$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x})$$

## SPECIAL LAGUERRE POLYNOMIALS

**28.3** $L_0(x) = 1$

**28.4** $L_1(x) = -x + 1$

**28.5** $L_2(x) = x^2 - 4x + 2$

**28.6** $L_3(x) = -x^3 + 9x^2 - 18x + 6$

**28.7** $L_4(x) = x^4 - 16x^3 + 72x^2 - 96x + 24$

**28.8** $L_5(x) = -x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120$

**28.9** $L_6(x) = x^6 - 36x^5 + 450x^4 - 2400x^3 + 5400x^2 - 4320x + 720$

**28.10** $L_7(x) = -x^7 + 49x^6 - 882x^5 + 7350x^4 - 29{,}400x^3 + 52{,}920x^2 - 35{,}280x + 5040$

## GENERATING FUNCTION

**28.11** $$\frac{e^{-xt/1-t}}{1-t} = \sum_{n=0}^{\infty} \frac{L_n(x)\, t^n}{n!}$$

## RECURRENCE FORMULAS

**28.12** $$L_{n+1}(x) - (2n+1-x)\, L_n(x) + n^2 L_{n-1}(x) = 0$$

**28.13** $$L_n'(x) - n L_{n-1}'(x) + n L_{n-1}(x) = 0$$

**28.14** $$x L_n'(x) = n L_n(x) - n^2 L_{n-1}(x)$$

## ORTHOGONALITY OF LAGUERRE POLYNOMIALS

**28.15** $$\int_0^\infty e^{-x} L_m(x)\, L_n(x)\, dx = 0 \qquad m \neq n$$

**28.16** $$\int_0^\infty e^{-x} \{L_n(x)\}^2\, dx = (n!)^2$$

## ORTHOGONAL SERIES

**28.17** $$f(x) = A_0 L_0(x) + A_1 L_1(x) + A_2 L_2(x) + \cdots$$

where

**28.18** $$A_k = \frac{1}{(k!)^2} \int_0^\infty e^{-x} f(x)\, L_k(x)\, dx$$

## SPECIAL RESULTS

**28.19** $L_n(0) = n!$ **28.20** $\displaystyle\int_0^x L_n(t)\, dt = L_n(x) - \frac{L_{n+1}(x)}{n+1}$

**28.21** $$L_n(x) = (-1)^n \left\{ x^n - \frac{n^2 x^{n-1}}{1!} + \frac{n^2(n-1)^2 x^{n-2}}{2!} - \cdots (-1)^n n! \right\}$$

**28.22** $$\int_0^\infty x^p e^{-x} L_n(x)\, dx = \begin{cases} 0 & \text{if } p < n \\ (-1)^n (n!)^2 & \text{if } p = n \end{cases}$$

**28.23** $$\sum_{k=0}^{n} \frac{L_k(x)\, L_k(y)}{(k!)^2} = \frac{L_n(x)\, L_{n+1}(y) - L_{n+1}(x)\, L_n(y)}{(n!)^2\, (x-y)}$$

**28.24** $$\sum_{k=0}^{\infty} \frac{t^k L_k(x)}{(k!)^2} = e^t J_0(2\sqrt{xt}\,)$$

**28.25** $$L_n(x) = \int_0^\infty u^n e^{x-u} J_0(2\sqrt{xu}\,)\, du$$

# 29 ASSOCIATED LAGUERRE POLYNOMIALS

## LAGUERRE'S ASSOCIATED DIFFERENTIAL EQUATION

**29.1** $$xy'' + (m+1-x)y' + (n-m)y = 0$$

## ASSOCIATED LAGUERRE POLYNOMIALS

Solutions of 29.1 for nonnegative integers $m$ and $n$ are given by the associated Laguerre polynomials

**29.2** $$L_n^m(x) = \frac{d^m}{dx^m} L_n(x)$$

where $L_n(x)$ are Laguerre polynomials [see page 153].

**29.3** $$L_n^0(x) = L_n(x)$$

**29.4** $$L_n^m(x) = 0 \qquad \text{if } m > n$$

## SPECIAL ASSOCIATED LAGUERRE POLYNOMIALS

**29.5** $L_1^1(x) = -1$

**29.6** $L_2^1(x) = 2x - 4$

**29.7** $L_2^2(x) = 2$

**29.8** $L_3^1(x) = -3x^2 + 18x - 18$

**29.9** $L_3^2(x) = -6x + 18$

**29.10** $L_3^3(x) = -6$

**29.11** $L_4^1(x) = 4x^3 - 48x^2 + 144x - 96$

**29.12** $L_4^2(x) = 12x^2 - 96x + 144$

**29.13** $L_4^3(x) = 24x - 96$

**29.14** $L_4^4(x) = 24$

## GENERATING FUNCTION FOR $L_n^m(x)$

**29.15** $$\frac{(-1)^m t^m}{(1-t)^{m+1}} e^{-xt/(1-t)} = \sum_{n=m}^{\infty} \frac{L_n^m(x)}{n!} t^n$$

## RECURRENCE FORMULAS

**29.16** $$\frac{n-m+1}{n+1} L_{n+1}^{m}(x) + (x+m-2n-1) L_n^m(x) + n^2 L_{n-1}^{m}(x) = 0$$

**29.17** $$\frac{d}{dx}\{L_n^m(x)\} = L_n^{m+1}(x)$$

**29.18** $$\frac{d}{dx}\{x^m e^{-x} L_n^m(x)\} = (m-n-1)x^{m-1}e^{-x} L_n^{m-1}(x)$$

**29.19** $$x\frac{d}{dx}\{L_n^m(x)\} = (x-m) L_n^m(x) + (m-n-1) L_n^{m-1}(x)$$

## ORTHOGONALITY

**29.20** $$\int_0^\infty x^m e^{-x} L_n^m(x) L_p^m(x)\,dx = 0 \qquad p \neq n$$

**29.21** $$\int_0^\infty x^m e^{-x} \{L_n^m(x)\}^2\,dx = \frac{(n!)^3}{(n-m)!}$$

## ORTHOGONAL SERIES

**29.22** $$f(x) = A_m L_m^m(x) + A_{m+1} L_{m+1}^m(x) + A_{m+2} L_{m+2}^m(x) + \cdots$$

where

**29.23** $$A_k = \frac{(k-m)!}{(k!)^3} \int_0^\infty x^m e^{-x} L_k^m(x) f(x)\,dx$$

## SPECIAL RESULTS

**29.24**

$$L_n^m(x) = (-1)^n \frac{n!}{(n-m)!}\left\{x^{n-m} - \frac{n(n-m)}{1!}x^{n-m-1} + \frac{n(n-1)(n-m)(n-m-1)}{2!}x^{n-m-2} + \cdots\right\}$$

**29.25** $$\int_0^\infty x^{m+1}e^{-x}\{L_n^m(x)\}^2\,dx = \frac{(2n-m+1)(n!)^3}{(n-m)!}$$

# 30 CHEBYSHEV POLYNOMIALS

## CHEBYSHEV'S DIFFERENTIAL EQUATION

**30.1** $$(1-x^2)y'' - xy' + n^2y = 0 \qquad n = 0, 1, 2, \ldots$$

## CHEBYSHEV POLYNOMIALS OF THE FIRST KIND

Solutions of 30.1 are given by

**30.2** $$T_n(x) = \cos(n \cos^{-1} x) = x^n - \binom{n}{2} x^{n-2}(1-x^2) + \binom{n}{4} x^{n-4}(1-x^2)^2 - \cdots$$

## SPECIAL CHEBYSHEV POLYNOMIALS OF THE FIRST KIND

**30.3** $T_0(x) = 1$

**30.4** $T_1(x) = x$

**30.5** $T_2(x) = 2x^2 - 1$

**30.6** $T_3(x) = 4x^3 - 3x$

**30.7** $T_4(x) = 8x^4 - 8x^2 + 1$

**30.8** $T_5(x) = 16x^5 - 20x^3 + 5x$

**30.9** $T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$

**30.10** $T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$

## GENERATING FUNCTION FOR $T_n(x)$

**30.11** $$\frac{1 - tx}{1 - 2tx + t^2} = \sum_{n=0}^{\infty} T_n(x)\, t^n$$

## SPECIAL VALUES

**30.12** $T_n(-x) = (-1)^n T_n(x)$

**30.13** $T_n(1) = 1$

**30.14** $T_n(-1) = (-1)^n$

**30.15** $T_{2n}(0) = (-1)^n$

**30.16** $T_{2n+1}(0) = 0$

## RECURSION FORMULA FOR $T_n(x)$

**30.17**
$$T_{n+1}(x) - 2x\,T_n(x) + T_{n-1}(x) = 0$$

## ORTHOGONALITY

**30.18**
$$\int_{-1}^{1} \frac{T_m(x)\,T_n(x)}{\sqrt{1-x^2}}\,dx = 0 \qquad m \neq n$$

**30.19**
$$\int_{-1}^{1} \frac{\{T_n(x)\}^2}{\sqrt{1-x^2}}\,dx = \begin{cases} \pi & \text{if } n = 0 \\ \pi/2 & \text{if } n = 1, 2, \ldots \end{cases}$$

## ORTHOGONAL SERIES

**30.20**
$$f(x) = \tfrac{1}{2}A_0\,T_0(x) + A_1\,T_1(x) + A_2\,T_2(x) + \cdots$$

where

**30.21**
$$A_k = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)\,T_k(x)}{\sqrt{1-x^2}}\,dx$$

## CHEBYSHEV POLYNOMIALS OF THE SECOND KIND

**30.22**
$$\begin{aligned} U_n(x) &= \frac{\sin\{(n+1)\cos^{-1}x\}}{\sin(\cos^{-1}x)} \\ &= \binom{n+1}{1}x^n - \binom{n+1}{3}x^{n-2}(1-x^2) + \binom{n+1}{5}x^{n-4}(1-x^2)^2 - \cdots \end{aligned}$$

## SPECIAL CHEBYSHEV POLYNOMIALS OF THE SECOND KIND

**30.23** $U_0(x) = 1$

**30.24** $U_1(x) = 2x$

**30.25** $U_2(x) = 4x^2 - 1$

**30.26** $U_3(x) = 8x^3 - 4x$

**30.27** $U_4(x) = 16x^4 - 12x^2 + 1$

**30.28** $U_5(x) = 32x^5 - 32x^3 + 6x$

**30.29** $U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1$

**30.30** $U_7(x) = 128x^7 - 192x^5 + 80x^3 - 8x$

## GENERATING FUNCTION FOR $U_n(x)$

**30.31**
$$\frac{1}{1 - 2tx + t^2} = \sum_{n=0}^{\infty} U_n(x)\,t^n$$

## SPECIAL VALUES

**30.32** $U_n(-x) = (-1)^n U_n(x)$

**30.33** $U_n(1) = n+1$

**30.34** $U_n(-1) = (-1)^n(n+1)$

**30.35** $U_{2n}(0) = (-1)^n$

**30.36** $U_{2n+1}(0) = 0$

## RECURSION FORMULA FOR $U_n(x)$

**30.37**
$$U_{n+1}(x) - 2x\,U_n(x) + U_{n-1}(x) = 0$$

## ORTHOGONALITY

**30.38**
$$\int_{-1}^{1} \sqrt{1-x^2}\,U_m(x)\,U_n(x)\,dx = 0 \qquad m \neq n$$

**30.39**
$$\int_{-1}^{1} \sqrt{1-x^2}\,\{U_n(x)\}^2\,dx = \frac{\pi}{2}$$

## ORTHOGONAL SERIES

**30.40**
$$f(x) = A_0 U_0(x) + A_1 U_1(x) + A_2 U_2(x) + \cdots$$

where

**30.41**
$$A_k = \frac{2}{\pi}\int_{-1}^{1} \sqrt{1-x^2}\,f(x)\,U_k(x)\,dx$$

## RELATIONSHIPS BETWEEN $T_n(x)$ AND $U_n(x)$

**30.42**
$$T_n(x) = U_n(x) - x\,U_{n-1}(x)$$

**30.43**
$$(1-x^2)\,U_{n-1}(x) = x\,T_n(x) - T_{n+1}(x)$$

**30.44**
$$U_n(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{T_{n+1}(v)\,dv}{(v-x)\sqrt{1-v^2}}$$

**30.45**
$$T_n(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1-v^2}\,U_{n-1}(v)}{x-v}\,dv$$

## GENERAL SOLUTION OF CHEBYSHEV'S DIFFERENTIAL EQUATION

**30.46**
$$y = \begin{cases} A\,T_n(x) + B\sqrt{1-x^2}\,U_{n-1}(x) & \text{if } n = 1, 2, 3, \ldots \\ A + B\sin^{-1}x & \text{if } n = 0 \end{cases}$$

# 31

# HYPERGEOMETRIC FUNCTIONS

## HYPERGEOMETRIC DIFFERENTIAL EQUATION

**31.1** $$x(1-x)y'' + \{c - (a+b+1)x\}y' - aby = 0$$

## HYPERGEOMETRIC FUNCTIONS

A solution of 31.1 is given by

**31.2** $$F(a, b; c; x) = 1 + \frac{a \cdot b}{1 \cdot c}x + \frac{a(a+1)b(b+1)}{1 \cdot 2 \cdot c(c+1)}x^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1 \cdot 2 \cdot 3 \cdot c(c+1)(c+2)}x^3 + \cdots$$

If $a, b, c$ are real, then the series converges for $-1 < x < 1$ provided that $c - (a+b) > -1$.

## SPECIAL CASES

**31.3** $F(-p, 1; 1; -x) = (1+x)^p$

**31.4** $F(1, 1; 2; -x) = [\ln(1+x)]/x$

**31.5** $\lim\limits_{n \to \infty} F(1, n; 1; x/n) = e^x$

**31.6** $F(\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}; \sin^2 x) = \cos x$

**31.7** $F(\frac{1}{2}, 1; 1; \sin^2 x) = \sec x$

**31.8** $F(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2) = (\sin^{-1} x)/x$

**31.9** $F(\frac{1}{2}, 1; \frac{3}{2}; -x^2) = (\tan^{-1} x)/x$

**31.10** $F(1, p; p; x) = 1/(1-x)$

**31.11** $F(n+1, -n; 1; (1-x)/2) = P_n(x)$

**31.12** $F(n, -n; \frac{1}{2}; (1-x)/2) = T_n(x)$

## GENERAL SOLUTION OF THE HYPERGEOMETRIC EQUATION

If $c$, $a-b$ and $c-a-b$ are all nonintegers, the general solution valid for $|x| < 1$ is

**31.13** $$y = A\,F(a, b; c; x) + Bx^{1-c}F(a-c+1, b-c+1; 2-c; x)$$

## MISCELLANEOUS PROPERTIES

**31.14** $$F(a, b; c; 1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}$$

**31.15** $$\frac{d}{dx}F(a, b; c; x) = \frac{ab}{c}F(a+1, b+1; c+1; x)$$

**31.16** $$F(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 u^{b-1}(1-u)^{c-b-1}(1-ux)^{-a}\,du$$

**31.17** $$F(a, b; c; x) = (1-x)^{c-a-b}F(c-a, c-b; c; x)$$

# 32 LAPLACE TRANSFORMS

## DEFINITION OF THE LAPLACE TRANSFORM OF $F(t)$

**32.1** $$\mathcal{L}\{F(t)\} \;=\; \int_0^\infty e^{-st}F(t)\,dt \;=\; f(s)$$

In general $f(s)$ will exist for $s > \alpha$ where $\alpha$ is some constant. $\mathcal{L}$ is called the *Laplace transform operator.*

## DEFINITION OF THE INVERSE LAPLACE TRANSFORM OF $f(s)$

If $\mathcal{L}\{F(t)\} = f(s)$, then we say that $F(t) = \mathcal{L}^{-1}\{f(s)\}$ is the *inverse Laplace transform* of $f(s)$. $\mathcal{L}^{-1}$ is called the *inverse Laplace transform operator.*

## COMPLEX INVERSION FORMULA

The inverse Laplace transform of $f(s)$ can be found directly by methods of complex variable theory. The result is

**32.2** $$F(t) \;=\; \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}f(s)\,ds \;=\; \frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT} e^{st}f(s)\,ds$$

where $c$ is chosen so that all the singular points of $f(s)$ lie to the left of the line $\operatorname{Re}\{s\} = c$ in the complex $s$ plane.

## TABLE OF GENERAL PROPERTIES OF LAPLACE TRANSFORMS

| | $f(s)$ | $F(t)$ |
|---|---|---|
| **32.3** | $a f_1(s) + b f_2(s)$ | $a F_1(t) + b F_2(t)$ |
| **32.4** | $f(s/a)$ | $a F(at)$ |
| **32.5** | $f(s-a)$ | $e^{at} F(t)$ |
| **32.6** | $e^{-as} f(s)$ | $\mathcal{U}(t-a) = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$ |
| **32.7** | $s f(s) - F(0)$ | $F'(t)$ |
| **32.8** | $s^2 f(s) - s F(0) - F'(0)$ | $F''(t)$ |
| **32.9** | $s^n f(s) - s^{n-1} F(0) - s^{n-2} F'(0) - \cdots - F^{(n-1)}(0)$ | $F^{(n)}(t)$ |
| **32.10** | $f'(s)$ | $-t F(t)$ |
| **32.11** | $f''(s)$ | $t^2 F(t)$ |
| **32.12** | $f^{(n)}(s)$ | $(-1)^n t^n F(t)$ |
| **32.13** | $\dfrac{f(s)}{s}$ | $\int_0^t F(u)\,du$ |
| **32.14** | $\dfrac{f(s)}{s^n}$ | $\int_0^t \cdots \int_0^t F(u)\,du^n = \int_0^t \dfrac{(t-u)^{n-1}}{(n-1)!} F(u)\,du$ |
| **32.15** | $f(s)\,g(s)$ | $\int_0^t F(u)\,G(t-u)\,du$ |

| | $f(s)$ | $F(t)$ |
|---|---|---|
| **32.16** | $\int_s^\infty f(u)\,du$ | $\frac{F(t)}{t}$ |
| **32.17** | $\frac{1}{1-e^{-sT}}\int_0^T e^{-su}F(u)\,du$ | $F(t) = F(t+T)$ |
| **32.18** | $\frac{f(\sqrt{s})}{s}$ | $\frac{1}{\sqrt{\pi t}}\int_0^\infty e^{-u^2/4t}F(u)\,du$ |
| **32.19** | $\frac{1}{s}f(1/s)$ | $\int_0^\infty J_0(2\sqrt{ut})\,F(u)\,du$ |
| **32.20** | $\frac{1}{s^{n+1}}f(1/s)$ | $t^{n/2}\int_0^\infty u^{-n/2}J_n(2\sqrt{ut})\,F(u)\,du$ |
| **32.21** | $\frac{f(s+1/s)}{s^2+1}$ | $\int_0^t J_0(2\sqrt{u(t-u)}\,)\,F(u)\,du$ |
| **32.22** | $\frac{1}{2\sqrt{\pi}}\int_0^\infty u^{-3/2}e^{-s^2/4u}f(u)\,du$ | $F(t^2)$ |
| **32.23** | $\frac{f(\ln s)}{s\ln s}$ | $\int_0^\infty \frac{t^u F(u)}{\Gamma(u+1)}\,du$ |
| **32.24** | $\frac{P(s)}{Q(s)}$<br>$P(s)$ = polynomial of degree less than $n$,<br>$Q(s) = (s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_n)$<br>where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all distinct. | $\sum_{k=1}^{n}\frac{P(\alpha_k)}{Q'(\alpha_k)}e^{\alpha_k t}$ |

## TABLE OF SPECIAL LAPLACE TRANSFORMS

| | $f(s)$ | $F(t)$ |
|---|---|---|
| **32.25** | $\frac{1}{s}$ | $1$ |
| **32.26** | $\frac{1}{s^2}$ | $t$ |
| **32.27** | $\frac{1}{s^n} \quad n = 1,2,3,\ldots$ | $\frac{t^{n-1}}{(n-1)!}, \quad 0! = 1$ |
| **32.28** | $\frac{1}{s^n} \quad n > 0$ | $\frac{t^{n-1}}{\Gamma(n)}$ |
| **32.29** | $\frac{1}{s-a}$ | $e^{at}$ |
| **32.30** | $\frac{1}{(s-a)^n} \quad n = 1,2,3,\ldots$ | $\frac{t^{n-1}e^{at}}{(n-1)!}, \quad 0! = 1$ |
| **32.31** | $\frac{1}{(s-a)^n} \quad n > 0$ | $\frac{t^{n-1}e^{at}}{\Gamma(n)}$ |
| **32.32** | $\frac{1}{s^2+a^2}$ | $\frac{\sin at}{a}$ |
| **32.33** | $\frac{s}{s^2+a^2}$ | $\cos at$ |
| **32.34** | $\frac{1}{(s-b)^2+a^2}$ | $\frac{e^{bt}\sin at}{a}$ |
| **32.35** | $\frac{s-b}{(s-b)^2+a^2}$ | $e^{bt}\cos at$ |
| **32.36** | $\frac{1}{s^2-a^2}$ | $\frac{\sinh at}{a}$ |
| **32.37** | $\frac{s}{s^2-a^2}$ | $\cosh at$ |
| **32.38** | $\frac{1}{(s-b)^2-a^2}$ | $\frac{e^{bt}\sinh at}{a}$ |

| | $f(s)$ | $F(t)$ |
|---|---|---|
| **32.39** | $\dfrac{s-b}{(s-b)^2-a^2}$ | $e^{bt}\cosh at$ |
| **32.40** | $\dfrac{1}{(s-a)(s-b)} \quad a \neq b$ | $\dfrac{e^{bt}-e^{at}}{b-a}$ |
| **32.41** | $\dfrac{s}{(s-a)(s-b)} \quad a \neq b$ | $\dfrac{be^{bt}-ae^{at}}{b-a}$ |
| **32.42** | $\dfrac{1}{(s^2+a^2)^2}$ | $\dfrac{\sin at - at\cos at}{2a^3}$ |
| **32.43** | $\dfrac{s}{(s^2+a^2)^2}$ | $\dfrac{t\sin at}{2a}$ |
| **32.44** | $\dfrac{s^2}{(s^2+a^2)^2}$ | $\dfrac{\sin at + at\cos at}{2a}$ |
| **32.45** | $\dfrac{s^3}{(s^2+a^2)^2}$ | $\cos at - \frac{1}{2}at\sin at$ |
| **32.46** | $\dfrac{s^2-a^2}{(s^2+a^2)^2}$ | $t\cos at$ |
| **32.47** | $\dfrac{1}{(s^2-a^2)^2}$ | $\dfrac{at\cosh at - \sinh at}{2a^3}$ |
| **32.48** | $\dfrac{s}{(s^2-a^2)^2}$ | $\dfrac{t\sinh at}{2a}$ |
| **32.49** | $\dfrac{s^2}{(s^2-a^2)^2}$ | $\dfrac{\sinh at + at\cosh at}{2a}$ |
| **32.50** | $\dfrac{s^3}{(s^2-a^2)^2}$ | $\cosh at + \frac{1}{2}at\sinh at$ |
| **32.51** | $\dfrac{s^2+a^2}{(s^2-a^2)^2}$ | $t\cosh at$ |
| **32.52** | $\dfrac{1}{(s^2+a^2)^3}$ | $\dfrac{(3-a^2t^2)\sin at - 3at\cos at}{8a^5}$ |
| **32.53** | $\dfrac{s}{(s^2+a^2)^3}$ | $\dfrac{t\sin at - at^2\cos at}{8a^3}$ |
| **32.54** | $\dfrac{s^2}{(s^2+a^2)^3}$ | $\dfrac{(1+a^2t^2)\sin at - at\cos at}{8a^3}$ |
| **32.55** | $\dfrac{s^3}{(s^2+a^2)^3}$ | $\dfrac{3t\sin at + at^2\cos at}{8a}$ |

| | $f(s)$ | $F(t)$ |
|---|---|---|
| **32.56** | $\dfrac{s^4}{(s^2+a^2)^3}$ | $\dfrac{(3-a^2t^2)\sin at + 5at\cos at}{8a}$ |
| **32.57** | $\dfrac{s^5}{(s^2+a^2)^3}$ | $\dfrac{(8-a^2t^2)\cos at - 7at\sin at}{8}$ |
| **32.58** | $\dfrac{3s^2-a^2}{(s^2+a^2)^3}$ | $\dfrac{t^2\sin at}{2a}$ |
| **32.59** | $\dfrac{s^3-3a^2s}{(s^2+a^2)^3}$ | $\frac{1}{2}t^2\cos at$ |
| **32.60** | $\dfrac{s^4-6a^2s^2+a^4}{(s^2+a^2)^4}$ | $\frac{1}{6}t^3\cos at$ |
| **32.61** | $\dfrac{s^3-a^2s}{(s^2+a^2)^4}$ | $\dfrac{t^3\sin at}{24a}$ |
| **32.62** | $\dfrac{1}{(s^2-a^2)^3}$ | $\dfrac{(3+a^2t^2)\sinh at - 3at\cosh at}{8a^5}$ |
| **32.63** | $\dfrac{s}{(s^2-a^2)^3}$ | $\dfrac{at^2\cosh at - t\sinh at}{8a^3}$ |
| **32.64** | $\dfrac{s^2}{(s^2-a^2)^3}$ | $\dfrac{at\cosh at + (a^2t^2-1)\sinh at}{8a^3}$ |
| **32.65** | $\dfrac{s^3}{(s^2-a^2)^3}$ | $\dfrac{3t\sinh at + at^2\cosh at}{8a}$ |
| **32.66** | $\dfrac{s^4}{(s^2-a^2)^3}$ | $\dfrac{(3+a^2t^2)\sinh at + 5at\cosh at}{8a}$ |
| **32.67** | $\dfrac{s^5}{(s^2-a^2)^3}$ | $\dfrac{(8+a^2t^2)\cosh at + 7at\sinh at}{8}$ |
| **32.68** | $\dfrac{3s^2+a^2}{(s^2-a^2)^3}$ | $\dfrac{t^2\sinh at}{2a}$ |
| **32.69** | $\dfrac{s^3+3a^2s}{(s^2-a^2)^3}$ | $\frac{1}{2}t^2\cosh at$ |
| **32.70** | $\dfrac{s^4+6a^2s^2+a^4}{(s^2-a^2)^4}$ | $\frac{1}{6}t^3\cosh at$ |
| **32.71** | $\dfrac{s^3+a^2s}{(s^2-a^2)^4}$ | $\dfrac{t^3\sinh at}{24a}$ |
| **32.72** | $\dfrac{1}{s^3+a^3}$ | $\dfrac{e^{at/2}}{3a^2}\left\{\sqrt{3}\sin\dfrac{\sqrt{3}\,at}{2} - \cos\dfrac{\sqrt{3}\,at}{2} + e^{-3at/2}\right\}$ |

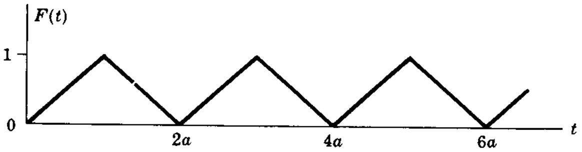
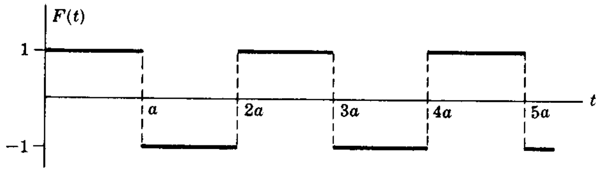
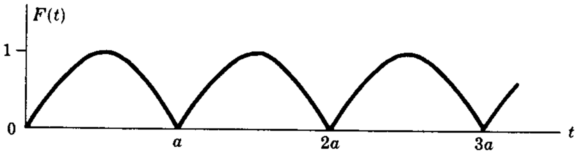
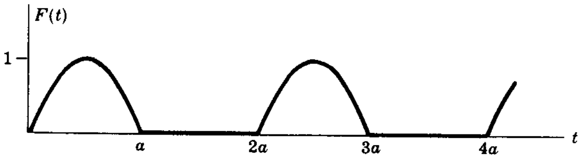
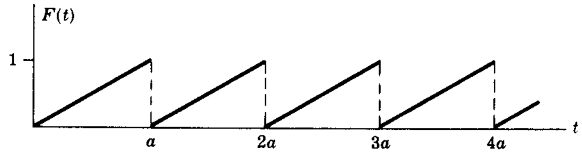
| | $f(s)$ | $F(t)$ |
|---|---|---|
| **32.73** | $\dfrac{s}{s^3+a^3}$ | $\dfrac{e^{at/2}}{3a}\left\{\cos\dfrac{\sqrt{3}\,at}{2} + \sqrt{3}\sin\dfrac{\sqrt{3}\,at}{2} - e^{-3at/2}\right\}$ |
| **32.74** | $\dfrac{s^2}{s^3+a^3}$ | $\dfrac{1}{3}\left(e^{-at} + 2e^{at/2}\cos\dfrac{\sqrt{3}\,at}{2}\right)$ |
| **32.75** | $\dfrac{1}{s^3-a^3}$ | $\dfrac{e^{-at/2}}{3a^2}\left\{e^{3at/2} - \cos\dfrac{\sqrt{3}\,at}{2} - \sqrt{3}\sin\dfrac{\sqrt{3}\,at}{2}\right\}$ |
| **32.76** | $\dfrac{s}{s^3-a^3}$ | $\dfrac{e^{-at/2}}{3a}\left\{\sqrt{3}\sin\dfrac{\sqrt{3}\,at}{2} - \cos\dfrac{\sqrt{3}\,at}{2} + e^{3at/2}\right\}$ |
| **32.77** | $\dfrac{s^2}{s^3-a^3}$ | $\dfrac{1}{3}\left(e^{at} + 2e^{-at/2}\cos\dfrac{\sqrt{3}\,at}{2}\right)$ |
| **32.78** | $\dfrac{1}{s^4+4a^4}$ | $\dfrac{1}{4a^3}(\sin at\cosh at - \cos at\sinh at)$ |
| **32.79** | $\dfrac{s}{s^4+4a^4}$ | $\dfrac{\sin at\sinh at}{2a^2}$ |
| **32.80** | $\dfrac{s^2}{s^4+4a^4}$ | $\dfrac{1}{2a}(\sin at\cosh at + \cos at\sinh at)$ |
| **32.81** | $\dfrac{s^3}{s^4+4a^4}$ | $\cos at\cosh at$ |
| **32.82** | $\dfrac{1}{s^4-a^4}$ | $\dfrac{1}{2a^3}(\sinh at - \sin at)$ |
| **32.83** | $\dfrac{s}{s^4-a^4}$ | $\dfrac{1}{2a^2}(\cosh at - \cos at)$ |
| **32.84** | $\dfrac{s^2}{s^4-a^4}$ | $\dfrac{1}{2a}(\sinh at + \sin at)$ |
| **32.85** | $\dfrac{s^3}{s^4-a^4}$ | $\frac{1}{2}(\cosh at + \cos at)$ |
| **32.86** | $\dfrac{1}{\sqrt{s+a}+\sqrt{s+b}}$ | $\dfrac{e^{-bt}-e^{-at}}{2(b-a)\sqrt{\pi t^3}}$ |
| **32.87** | $\dfrac{1}{s\sqrt{s+a}}$ | $\dfrac{\operatorname{erf}\sqrt{at}}{\sqrt{a}}$ |
| **32.88** | $\dfrac{1}{\sqrt{s}\,(s-a)}$ | $\dfrac{e^{at}\operatorname{erf}\sqrt{at}}{\sqrt{a}}$ |
| **32.89** | $\dfrac{1}{\sqrt{s-a}+b}$ | $e^{at}\left\{\dfrac{1}{\sqrt{\pi t}} - b\,e^{b^2t}\operatorname{erfc}(b\sqrt{t})\right\}$ |

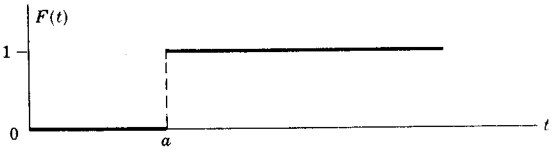
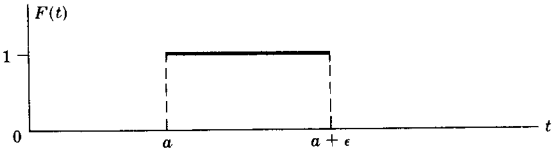
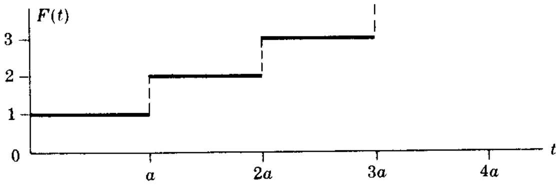
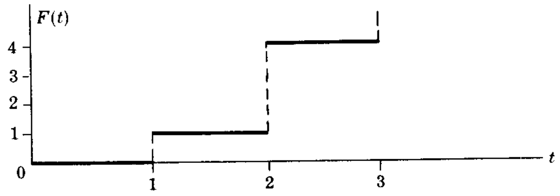
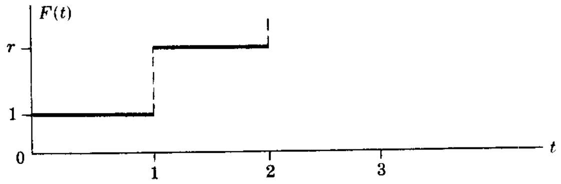
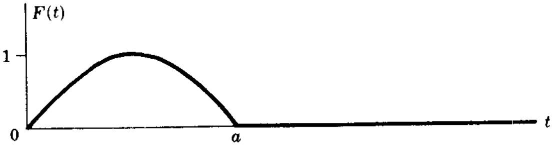
| | $f(s)$ | $F(t)$ |
|---|---|---|
| **32.90** | $\dfrac{1}{\sqrt{s^2+a^2}}$ | $J_0(at)$ |
| **32.91** | $\dfrac{1}{\sqrt{s^2-a^2}}$ | $I_0(at)$ |
| **32.92** | $\dfrac{(\sqrt{s^2+a^2}-s)^n}{\sqrt{s^2+a^2}} \quad n>-1$ | $a^n J_n(at)$ |
| **32.93** | $\dfrac{(s-\sqrt{s^2-a^2})^n}{\sqrt{s^2-a^2}} \quad n>-1$ | $a^n I_n(at)$ |
| **32.94** | $\dfrac{e^{b(s-\sqrt{s^2+a^2})}}{\sqrt{s^2+a^2}}$ | $J_0(a\sqrt{t(t+2b)})$ |
| **32.95** | $\dfrac{e^{-b\sqrt{s^2+a^2}}}{\sqrt{s^2+a^2}}$ | $\begin{cases} J_0(a\sqrt{t^2-b^2}) & t>b \\ 0 & t<b \end{cases}$ |
| **32.96** | $\dfrac{1}{(s^2+a^2)^{3/2}}$ | $\dfrac{tJ_1(at)}{a}$ |
| **32.97** | $\dfrac{s}{(s^2+a^2)^{3/2}}$ | $tJ_0(at)$ |
| **32.98** | $\dfrac{s^2}{(s^2+a^2)^{3/2}}$ | $J_0(at) - atJ_1(at)$ |
| **32.99** | $\dfrac{1}{(s^2-a^2)^{3/2}}$ | $\dfrac{tI_1(at)}{a}$ |
| **32.100** | $\dfrac{s}{(s^2-a^2)^{3/2}}$ | $tI_0(at)$ |
| **32.101** | $\dfrac{s^2}{(s^2-a^2)^{3/2}}$ | $I_0(at) + atI_1(at)$ |
| **32.102** | $\dfrac{1}{s(e^s-1)} = \dfrac{e^{-s}}{s(1-e^{-s})}$<br>See also entry 32.165. | $F(t)=n, \; n \leqq t < n+1, \; n=0,1,2,\ldots$ |
| **32.103** | $\dfrac{1}{s(e^s-r)} = \dfrac{e^{-s}}{s(1-re^{-s})}$ | $F(t) = \sum_{k=1}^{[t]} r^k$<br>where $[t]$ = greatest integer $\leqq t$ |
| **32.104** | $\dfrac{e^s-1}{s(e^s-r)} = \dfrac{1-e^{-s}}{s(1-re^{-s})}$<br>See also entry 32.167. | $F(t)=r^n, \; n \leqq t < n+1, \; n=0,1,2,\ldots$ |
| **32.105** | $\dfrac{e^{-a/s}}{\sqrt{s}}$ | $\dfrac{\cos 2\sqrt{at}}{\sqrt{\pi t}}$ |

| | $f(s)$ | $F(t)$ |
|---|---|---|
| **32.106** | $\dfrac{e^{-a/s}}{s^{3/2}}$ | $\dfrac{\sin 2\sqrt{at}}{\sqrt{\pi a}}$ |
| **32.107** | $\dfrac{e^{-a/s}}{s^{n+1}} \qquad n > -1$ | $\left(\dfrac{t}{a}\right)^{n/2} J_n(2\sqrt{at})$ |
| **32.108** | $\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}}$ | $\dfrac{e^{-a^2/4t}}{\sqrt{\pi t}}$ |
| **32.109** | $e^{-a\sqrt{s}}$ | $\dfrac{a}{2\sqrt{\pi t^3}}\, e^{-a^2/4t}$ |
| **32.110** | $\dfrac{1 - e^{-a\sqrt{s}}}{s}$ | $\operatorname{erf}(a/2\sqrt{t})$ |
| **32.111** | $\dfrac{e^{-a\sqrt{s}}}{s}$ | $\operatorname{erfc}(a/2\sqrt{t})$ |
| **32.112** | $\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}\,(\sqrt{s} + b)}$ | $e^{b(bt+a)} \operatorname{erfc}\left(b\sqrt{t} + \dfrac{a}{2\sqrt{t}}\right)$ |
| **32.113** | $\dfrac{e^{-a/\sqrt{s}}}{s^{n+1}} \qquad n > -1$ | $\dfrac{1}{\sqrt{\pi t}\, a^{2n+1}} \displaystyle\int_0^\infty u^n\, e^{-u^2/4a^2 t}\, J_{2n}(2\sqrt{u})\, du$ |
| **32.114** | $\ln\left(\dfrac{s+a}{s+b}\right)$ | $\dfrac{e^{-bt} - e^{-at}}{t}$ |
| **32.115** | $\dfrac{\ln\,[(s^2 + a^2)/a^2]}{2s}$ | $Ci(at)$ |
| **32.116** | $\dfrac{\ln\,[(s+a)/a]}{s}$ | $Ei(at)$ |
| **32.117** | $-\dfrac{(\gamma + \ln s)}{s}$<br>$\gamma$ = Euler's constant = .5772156... | $\ln t$ |
| **32.118** | $\ln\left(\dfrac{s^2 + a^2}{s^2 + b^2}\right)$ | $\dfrac{2(\cos at - \cos bt)}{t}$ |
| **32.119** | $\dfrac{\pi^2}{6s} + \dfrac{(\gamma + \ln s)^2}{s}$<br>$\gamma$ = Euler's constant = .5772156... | $\ln^2 t$ |
| **32.120** | $\dfrac{\ln s}{s}$ | $-(\ln t + \gamma)$<br>$\gamma$ = Euler's constant = .5772156... |
| **32.121** | $\dfrac{\ln^2 s}{s}$ | $(\ln t + \gamma)^2 - \frac{1}{6}\pi^2$<br>$\gamma$ = Euler's constant = .5772156... |

| | $f(s)$ | $F(t)$ |
|---|---|---|
| **32.122** | $\frac{\Gamma'(n+1)-\Gamma(n+1)\ln s}{s^{n+1}} \qquad n>-1$ | $t^n \ln t$ |
| **32.123** | $\tan^{-1}(a/s)$ | $\frac{\sin at}{t}$ |
| **32.124** | $\frac{\tan^{-1}(a/s)}{s}$ | $Si(at)$ |
| **32.125** | $\frac{e^{a/s}}{\sqrt{s}}\operatorname{erfc}(\sqrt{a/s})$ | $\frac{e^{-2\sqrt{at}}}{\sqrt{\pi t}}$ |
| **32.126** | $e^{s^2/4a^2}\operatorname{erfc}(s/2a)$ | $\frac{2a}{\sqrt{\pi}}e^{-a^2t^2}$ |
| **32.127** | $\frac{e^{s^2/4a^2}\operatorname{erfc}(s/2a)}{s}$ | $\operatorname{erf}(at)$ |
| **32.128** | $\frac{e^{as}\operatorname{erfc}\sqrt{as}}{\sqrt{s}}$ | $\frac{1}{\sqrt{\pi(t+a)}}$ |
| **32.129** | $e^{as}\,Ei(as)$ | $\frac{1}{t+a}$ |
| **32.130** | $\frac{1}{a}\left[\cos as\left\{\frac{\pi}{2}-Si(as)\right\}-\sin as\,Ci(as)\right]$ | $\frac{1}{t^2+a^2}$ |
| **32.131** | $\sin as\left\{\frac{\pi}{2}-Si(as)\right\}+\cos as\,Ci(as)$ | $\frac{t}{t^2+a^2}$ |
| **32.132** | $\frac{\cos as\left\{\frac{\pi}{2}-Si(as)\right\}-\sin as\,Ci(as)}{s}$ | $\tan^{-1}(t/a)$ |
| **32.133** | $\frac{\sin as\left\{\frac{\pi}{2}-Si(as)\right\}+\cos as\,Ci(as)}{s}$ | $\frac{1}{2}\ln\left(\frac{t^2+a^2}{a^2}\right)$ |
| **32.134** | $\left[\frac{\pi}{2}-Si(as)\right]^2+Ci^2(as)$ | $\frac{1}{t}\ln\left(\frac{t^2+a^2}{a^2}\right)$ |
| **32.135** | $0$ | $\mathcal{N}(t)$ = null function |
| **32.136** | $1$ | $\delta(t)$ = delta function |
| **32.137** | $e^{-as}$ | $\delta(t-a)$ |
| **32.138** | $\frac{e^{-as}}{s}$<br>See also entry 32.163. | $\mathcal{U}(t-a)$ |

| | $f(s)$ | $F(t)$ |
|---|---|---|
| **32.139** | $\dfrac{\sinh sx}{s \sinh sa}$ | $\dfrac{x}{a} + \dfrac{2}{\pi} \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n} \sin \dfrac{n\pi x}{a} \cos \dfrac{n\pi t}{a}$ |
| **32.140** | $\dfrac{\sinh sx}{s \cosh sa}$ | $\dfrac{4}{\pi} \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{2n-1} \sin \dfrac{(2n-1)\pi x}{2a} \sin \dfrac{(2n-1)\pi t}{2a}$ |
| **32.141** | $\dfrac{\cosh sx}{s \sinh as}$ | $\dfrac{t}{a} + \dfrac{2}{\pi} \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n} \cos \dfrac{n\pi x}{a} \sin \dfrac{n\pi t}{a}$ |
| **32.142** | $\dfrac{\cosh sx}{s \cosh sa}$ | $1 + \dfrac{4}{\pi} \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{2n-1} \cos \dfrac{(2n-1)\pi x}{2a} \cos \dfrac{(2n-1)\pi t}{2a}$ |
| **32.143** | $\dfrac{\sinh sx}{s^2 \sinh sa}$ | $\dfrac{xt}{a} + \dfrac{2a}{\pi^2} \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n^2} \sin \dfrac{n\pi x}{a} \sin \dfrac{n\pi t}{a}$ |
| **32.144** | $\dfrac{\sinh sx}{s^2 \cosh sa}$ | $x + \dfrac{8a}{\pi^2} \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{(2n-1)^2} \sin \dfrac{(2n-1)\pi x}{2a} \cos \dfrac{(2n-1)\pi t}{2a}$ |
| **32.145** | $\dfrac{\cosh sx}{s^2 \sinh sa}$ | $\dfrac{t^2}{2a} + \dfrac{2a}{\pi^2} \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n^2} \cos \dfrac{n\pi x}{a} \left(1 - \cos \dfrac{n\pi t}{a}\right)$ |
| **32.146** | $\dfrac{\cosh sx}{s^2 \cosh sa}$ | $t + \dfrac{8a}{\pi^2} \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{(2n-1)^2} \cos \dfrac{(2n-1)\pi x}{2a} \sin \dfrac{(2n-1)\pi t}{2a}$ |
| **32.147** | $\dfrac{\cosh sx}{s^3 \cosh sa}$ | $\frac{1}{2}(t^2 + x^2 - a^2) - \dfrac{16a^2}{\pi^3} \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{(2n-1)^3} \cos \dfrac{(2n-1)\pi x}{2a} \cos \dfrac{(2n-1)\pi t}{2a}$ |
| **32.148** | $\dfrac{\sinh x\sqrt{s}}{\sinh a\sqrt{s}}$ | $\dfrac{2\pi}{a^2} \sum\limits_{n=1}^{\infty} (-1)^n n\, e^{-n^2\pi^2 t/a^2} \sin \dfrac{n\pi x}{a}$ |
| **32.149** | $\dfrac{\cosh x\sqrt{s}}{\cosh a\sqrt{s}}$ | $\dfrac{\pi}{a^2} \sum\limits_{n=1}^{\infty} (-1)^{n-1} (2n-1)\, e^{-(2n-1)^2\pi^2 t/4a^2} \cos \dfrac{(2n-1)\pi x}{2a}$ |
| **32.150** | $\dfrac{\sinh x\sqrt{s}}{\sqrt{s} \cosh a\sqrt{s}}$ | $\dfrac{2}{a} \sum\limits_{n=1}^{\infty} (-1)^{n-1} e^{-(2n-1)^2\pi^2 t/4a^2} \sin \dfrac{(2n-1)\pi x}{2a}$ |
| **32.151** | $\dfrac{\cosh x\sqrt{s}}{\sqrt{s} \sinh a\sqrt{s}}$ | $\dfrac{1}{a} + \dfrac{2}{a} \sum\limits_{n=1}^{\infty} (-1)^n e^{-n^2\pi^2 t/a^2} \cos \dfrac{n\pi x}{a}$ |
| **32.152** | $\dfrac{\sinh x\sqrt{s}}{s \sinh a\sqrt{s}}$ | $\dfrac{x}{a} + \dfrac{2}{\pi} \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n} e^{-n^2\pi^2 t/a^2} \sin \dfrac{n\pi x}{a}$ |
| **32.153** | $\dfrac{\cosh x\sqrt{s}}{s \cosh a\sqrt{s}}$ | $1 + \dfrac{4}{\pi} \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{2n-1} e^{-(2n-1)^2\pi^2 t/4a^2} \cos \dfrac{(2n-1)\pi x}{2a}$ |
| **32.154** | $\dfrac{\sinh x\sqrt{s}}{s^2 \sinh a\sqrt{s}}$ | $\dfrac{xt}{a} + \dfrac{2a^2}{\pi^3} \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n^3} (1 - e^{-n^2\pi^2 t/a^2}) \sin \dfrac{n\pi x}{a}$ |
| **32.155** | $\dfrac{\cosh x\sqrt{s}}{s^2 \cosh a\sqrt{s}}$ | $\frac{1}{2}(x^2 - a^2) + t - \dfrac{16a^2}{\pi^3} \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{(2n-1)^3} e^{-(2n-1)^2\pi^2 t/4a^2} \cos \dfrac{(2n-1)\pi x}{2a}$ |

| | $f(s)$ | $F(t)$ |
|---|---|---|
| **32.156** | $\frac{J_0(ix\sqrt{s})}{s\,J_0(ia\sqrt{s})}$ | $1 - 2\sum_{n=1}^{\infty} \frac{e^{-\lambda_n^2 t/a^2} J_0(\lambda_n x/a)}{\lambda_n J_1(\lambda_n)}$<br>where $\lambda_1, \lambda_2, \ldots$ are the positive roots of $J_0(\lambda) = 0$ |
| **32.157** | $\frac{J_0(ix\sqrt{s})}{s^2\,J_0(ia\sqrt{s})}$ | $\frac{1}{4}(x^2 - a^2) + t + 2a^2 \sum_{n=1}^{\infty} \frac{e^{-\lambda_n^2 t/a^2} J_0(\lambda_n x/a)}{\lambda_n^3 J_1(\lambda_n)}$<br>where $\lambda_1, \lambda_2, \ldots$ are the positive roots of $J_0(\lambda) = 0$ |
| **32.158** | $\frac{1}{as^2} \tanh\left(\frac{as}{2}\right)$ | **Triangular wave function**<br> <br>**Fig. 32-1** |
| **32.159** | $\frac{1}{s} \tanh\left(\frac{as}{2}\right)$ | **Square wave function**<br> <br>**Fig. 32-2** |
| **32.160** | $\frac{\pi a}{a^2 s^2 + \pi^2} \coth\left(\frac{as}{2}\right)$ | **Rectified sine wave function**<br> <br>**Fig. 32-3** |
| **32.161** | $\frac{\pi a}{(a^2 s^2 + \pi^2)(1 - e^{-as})}$ | **Half rectified sine wave function**<br> <br>**Fig. 32-4** |
| **32.162** | $\frac{1}{as^2} - \frac{e^{-as}}{s(1 - e^{-as})}$ | **Saw tooth wave function**<br> <br>**Fig. 32-5** |

| | $f(s)$ | $F(t)$ |
|---|---|---|
| **32.163** | $\dfrac{e^{-as}}{s}$<br>See also entry 32.138. | **Heaviside's unit function $\mathcal{U}(t-a)$**<br> <br>**Fig. 32-6** |
| **32.164** | $\dfrac{e^{-as}(1-e^{-\epsilon s})}{s}$ | **Pulse function**<br> <br>**Fig. 32-7** |
| **32.165** | $\dfrac{1}{s(1-e^{-as})}$<br>See also entry 32.102. | **Step function**<br> <br>**Fig. 32-8** |
| **32.166** | $\dfrac{e^{-s}+e^{-2s}}{s(1-e^{-s})^2}$ | $F(t)=n^2,\ n \leqq t < n+1,\ n=0,1,2,\ldots$<br> <br>**Fig. 32-9** |
| **32.167** | $\dfrac{1-e^{-s}}{s(1-re^{-s})}$<br>See also entry 32.104. | $F(t)=r^n,\ n \leqq t < n+1,\ n=0,1,2,\ldots$<br> <br>**Fig. 32-10** |
| **32.168** | $\dfrac{\pi a(1+e^{-as})}{a^2s^2+\pi^2}$ | $F(t) = \begin{cases} \sin(\pi t/a) & 0 \leqq t \leqq a \\ 0 & t > a \end{cases}$<br> <br>**Fig. 32-11** |

# 33 FOURIER TRANSFORMS

## FOURIER'S INTEGRAL THEOREM

**33.1**

$$f(x) = \int_0^\infty \{A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\}\,d\alpha$$

where

**33.2**

$$\begin{cases} A(\alpha) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx \\ B(\alpha) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx \end{cases}$$

Sufficient conditions under which this theorem holds are:

(i) $f(x)$ and $f'(x)$ are piecewise continuous in every finite interval $-L < x < L$;

(ii) $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges;

(iii) $f(x)$ is replaced by $\frac{1}{2}\{f(x+0) + f(x-0)\}$ if $x$ is a point of discontinuity.

## EQUIVALENT FORMS OF FOURIER'S INTEGRAL THEOREM

**33.3**

$$f(x) = \frac{1}{2\pi}\int_{\alpha=-\infty}^{\infty}\int_{u=-\infty}^{\infty} f(u)\cos\alpha\,(x-u)\,du\,d\alpha$$

**33.4**

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x}\,d\alpha\int_{-\infty}^{\infty} f(u)\,e^{-i\alpha u}\,du \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\,e^{i\alpha(x-u)}\,du\,d\alpha \end{aligned}$$

**33.5**

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\alpha x\,d\alpha\int_0^\infty f(u)\sin\alpha u\,du$$

where $f(x)$ is an *odd function* $[f(-x) = -f(x)]$.

**33.6**

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\alpha x\,d\alpha\int_0^\infty f(u)\cos\alpha u\,du$$

where $f(x)$ is an *even function* $[f(-x) = f(x)]$.

## FOURIER TRANSFORMS

The Fourier transform of $f(x)$ is defined as

**33.7**
$$\mathcal{F}\{f(x)\} = F(\alpha) = \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$$

Then from 33.7 the inverse Fourier transform of $F(\alpha)$ is

**33.8**
$$\mathcal{F}^{-1}\{F(\alpha)\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\, e^{i\alpha x}\, d\alpha$$

We call $f(x)$ and $F(\alpha)$ *Fourier transform pairs.*

## CONVOLUTION THEOREM FOR FOURIER TRANSFORMS

If $F(\alpha) = \mathcal{F}\{f(x)\}$ and $G(\alpha) = \mathcal{F}\{g(x)\}$, then

**33.9**
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\, G(\alpha)\, e^{i\alpha x}\, d\alpha = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du = f{*}g$$

where $f{*}g$ is called the *convolution* of $f$ and $g$. Thus

**33.10**
$$\mathcal{F}\{f{*}g\} = \mathcal{F}\{f\}\,\mathcal{F}\{g\}$$

## PARSEVAL'S IDENTITY

If $F(\alpha) = \mathcal{F}\{f(x)\}$, then

**33.11**
$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\alpha)|^2\, d\alpha$$

More generally if $F(\alpha) = \mathcal{F}\{f(x)\}$ and $G(\alpha) = \mathcal{F}\{g(x)\}$, then

**33.12**
$$\int_{-\infty}^{\infty} f(x)\, \overline{g(x)}\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\, \overline{G(\alpha)}\, d\alpha$$

where the bar denotes complex conjugate.

## FOURIER SINE TRANSFORMS

The Fourier sine transform of $f(x)$ is defined as

**33.13**
$$F_S(\alpha) = \mathcal{F}_S\{f(x)\} = \int_0^{\infty} f(x) \sin \alpha x\, dx$$

Then from 33.13 the inverse Fourier sine transform of $F_S(\alpha)$ is

**33.14**
$$f(x) = \mathcal{F}_S^{-1}\{F_S(\alpha)\} = \frac{2}{\pi}\int_0^{\infty} F_S(\alpha) \sin \alpha x\, d\alpha$$

## FOURIER COSINE TRANSFORMS

The Fourier cosine transform of $f(x)$ is defined as

**33.15** $$F_C(\alpha) = \mathcal{F}_C\{f(x)\} = \int_0^\infty f(x)\cos\alpha x\, dx$$

Then from 33.15 the inverse Fourier cosine transform of $F_C(\alpha)$ is

**33.16** $$f(x) = \mathcal{F}_C^{-1}\{F_C(\alpha)\} = \frac{2}{\pi}\int_0^\infty F_C(\alpha)\cos\alpha x\, d\alpha$$

## SPECIAL FOURIER TRANSFORM PAIRS

| | $f(x)$ | $F(\alpha)$ |
|---|---|---|
| **33.17** | $\begin{cases} 1 & \lvert x\rvert < b \\ 0 & \lvert x\rvert > b \end{cases}$ | $\dfrac{2\sin b\alpha}{\alpha}$ |
| **33.18** | $\dfrac{1}{x^2+b^2}$ | $\dfrac{\pi e^{-b\alpha}}{b}$ |
| **33.19** | $\dfrac{x}{x^2+b^2}$ | $-\dfrac{\pi i\alpha}{b}e^{-b\alpha}$ |
| **33.20** | $f^{(n)}(x)$ | $i^n\alpha^nF(\alpha)$ |
| **33.21** | $x^nf(x)$ | $i^n\dfrac{d^nF}{d\alpha^n}$ |
| **33.22** | $f(bx)e^{itx}$ | $\dfrac{1}{b}F\left(\dfrac{\alpha-t}{b}\right)$ |

## SPECIAL FOURIER SINE TRANSFORMS

| | $f(x)$ | $F_C(\alpha)$ |
|---|---|---|
| **33.23** | $\begin{cases} 1 & 0 < x < b \\ 0 & x > b \end{cases}$ | $\dfrac{1 - \cos b\alpha}{\alpha}$ |
| **33.24** | $x^{-1}$ | $\dfrac{\pi}{2}$ |
| **33.25** | $\dfrac{x}{x^2 + b^2}$ | $\dfrac{\pi}{2} e^{-b\alpha}$ |
| **33.26** | $e^{-bx}$ | $\dfrac{\alpha}{\alpha^2 + b^2}$ |
| **33.27** | $x^{n-1} e^{-bx}$ | $\dfrac{\Gamma(n) \sin (n \tan^{-1} \alpha/b)}{(\alpha^2 + b^2)^{n/2}}$ |
| **33.28** | $xe^{-bx^2}$ | $\dfrac{\sqrt{\pi}}{4b^{3/2}} \alpha e^{-\alpha^2/4b}$ |
| **33.29** | $x^{-1/2}$ | $\sqrt{\dfrac{\pi}{2\alpha}}$ |
| **33.30** | $x^{-n}$ | $\dfrac{\pi\alpha^{n-1} \csc (n\pi/2)}{2\,\Gamma(n)} \quad 0 < n < 2$ |
| **33.31** | $\dfrac{\sin bx}{x}$ | $\dfrac{1}{2} \ln \left(\dfrac{\alpha + b}{\alpha - b}\right)$ |
| **33.32** | $\dfrac{\sin bx}{x^2}$ | $\begin{cases} \pi\alpha/2 & \alpha < b \\ \pi b/2 & \alpha > b \end{cases}$ |
| **33.33** | $\dfrac{\cos bx}{x}$ | $\begin{cases} 0 & \alpha < b \\ \pi/4 & \alpha = b \\ \pi/2 & \alpha > b \end{cases}$ |
| **33.34** | $\tan^{-1} (x/b)$ | $\dfrac{\pi}{2\alpha} e^{-b\alpha}$ |
| **33.35** | $\csc bx$ | $\dfrac{\pi}{2b} \tanh \dfrac{\pi\alpha}{2b}$ |
| **33.36** | $\dfrac{1}{e^{2x} - 1}$ | $\dfrac{\pi}{4} \coth \left(\dfrac{\pi\alpha}{2}\right) - \dfrac{1}{2\alpha}$ |

## SPECIAL FOURIER COSINE TRANSFORMS

| | $f(x)$ | $F_C(\alpha)$ |
|---|---|---|
| **33.37** | $\begin{cases} 1 & 0 < x < b \\ 0 & x > b \end{cases}$ | $\dfrac{\sin b\alpha}{\alpha}$ |
| **33.38** | $\dfrac{1}{x^2 + b^2}$ | $\dfrac{\pi e^{-b\alpha}}{2b}$ |
| **33.39** | $e^{-bx}$ | $\dfrac{b}{\alpha^2 + b^2}$ |
| **33.40** | $x^{n-1} e^{-bx}$ | $\dfrac{\Gamma(n)\cos(n\tan^{-1}\alpha/b)}{(\alpha^2 + b^2)^{n/2}}$ |
| **33.41** | $e^{-bx^2}$ | $\dfrac{1}{2}\sqrt{\dfrac{\pi}{b}}e^{-\alpha^2/4b}$ |
| **33.42** | $x^{-1/2}$ | $\sqrt{\dfrac{\pi}{2\alpha}}$ |
| **33.43** | $x^{-n}$ | $\dfrac{\pi\alpha^{n-1}\sec(n\pi/2)}{2\,\Gamma(n)}, \quad 0 < n < 1$ |
| **33.44** | $\ln\left(\dfrac{x^2 + b^2}{x^2 + c^2}\right)$ | $\dfrac{e^{-c\alpha} - e^{-b\alpha}}{\pi\alpha}$ |
| **33.45** | $\dfrac{\sin bx}{x}$ | $\begin{cases} \pi/2 & \alpha < b \\ \pi/4 & \alpha = b \\ 0 & \alpha > b \end{cases}$ |
| **33.46** | $\sin bx^2$ | $\sqrt{\dfrac{\pi}{8b}}\left(\cos\dfrac{\alpha^2}{4b} - \sin\dfrac{\alpha^2}{4b}\right)$ |
| **33.47** | $\cos bx^2$ | $\sqrt{\dfrac{\pi}{8b}}\left(\cos\dfrac{\alpha^2}{4b} + \sin\dfrac{\alpha^2}{4b}\right)$ |
| **33.48** | $\operatorname{sech} bx$ | $\dfrac{\pi}{2b}\operatorname{sech}\dfrac{\pi\alpha}{2b}$ |
| **33.49** | $\dfrac{\cosh(\sqrt{\pi}\,x/2)}{\cosh(\sqrt{\pi}\,x)}$ | $\sqrt{\dfrac{\pi}{2}}\,\dfrac{\cosh(\sqrt{\pi}\,\alpha/2)}{\cosh(\sqrt{\pi}\,\alpha)}$ |
| **33.50** | $\dfrac{e^{-b\sqrt{x}}}{\sqrt{x}}$ | $\sqrt{\dfrac{\pi}{2\alpha}}\{\cos(2b\sqrt{\alpha}) - \sin(2b\sqrt{\alpha})\}$ |

# 34 ELLIPTIC FUNCTIONS

## INCOMPLETE ELLIPTIC INTEGRAL OF THE FIRST KIND

**34.1** $$u = F(k,\phi) = \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{x} \frac{dv}{\sqrt{(1-v^2)(1-k^2v^2)}}$$

where $\phi = \operatorname{am} u$ is called the *amplitude* of $u$ and $x = \sin\phi$, and where here and below $0 < k < 1$.

## COMPLETE ELLIPTIC INTEGRAL OF THE FIRST KIND

**34.2** $$K = F(k,\pi/2) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{1} \frac{dv}{\sqrt{(1-v^2)(1-k^2v^2)}}$$
$$= \frac{\pi}{2}\left\{1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \cdots\right\}$$

## INCOMPLETE ELLIPTIC INTEGRAL OF THE SECOND KIND

**34.3** $$E(k,\phi) = \int_0^{\phi} \sqrt{1-k^2\sin^2\theta}\, d\theta = \int_0^{x} \frac{\sqrt{1-k^2v^2}}{\sqrt{1-v^2}}\, dv$$

## COMPLETE ELLIPTIC INTEGRAL OF THE SECOND KIND

**34.4** $$E = E(k,\pi/2) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\, d\theta = \int_0^{1} \frac{\sqrt{1-k^2v^2}}{\sqrt{1-v^2}}\, dv$$
$$= \frac{\pi}{2}\left\{1 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{k^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \frac{k^6}{5} - \cdots\right\}$$

## INCOMPLETE ELLIPTIC INTEGRAL OF THE THIRD KIND

**34.5** $$\Pi(k,n,\phi) = \int_0^{\phi} \frac{d\theta}{(1+n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} = \int_0^{x} \frac{dv}{(1+nv^2)\sqrt{(1-v^2)(1-k^2v^2)}}$$

## COMPLETE ELLIPTIC INTEGRAL OF THE THIRD KIND

**34.6** $$\Pi(k, n, \pi/2) = \int_0^{\pi/2} \frac{d\theta}{(1 + n \sin^2\theta)\sqrt{1 - k^2 \sin^2\theta}} = \int_0^1 \frac{dv}{(1 + nv^2)\sqrt{(1 - v^2)(1 - k^2v^2)}}$$

## LANDEN'S TRANSFORMATION

**34.7** $$\tan\phi = \frac{\sin 2\phi_1}{k + \cos 2\phi_1} \quad \text{or} \quad k \sin\phi = \sin(2\phi_1 - \phi)$$

This yields

**34.8** $$F(k, \phi) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} = \frac{2}{1 + k}\int_0^{\phi_1} \frac{d\theta_1}{\sqrt{1 - k_1^2 \sin^2\theta_1}}$$

where $k_1 = 2\sqrt{k}/(1 + k)$. By successive applications, sequences $k_1, k_2, k_3, \ldots$ and $\phi_1, \phi_2, \phi_3, \ldots$ are obtained such that $k < k_1 < k_2 < k_3 < \cdots < 1$ where $\lim\limits_{n\to\infty} k_n = 1$. It follows that

**34.9** $$F(k, \Phi) = \sqrt{\frac{k_1k_2k_3\ldots}{k}} \int_0^{\Phi} \frac{d\theta}{\sqrt{1 - \sin^2\theta}} = \sqrt{\frac{k_1k_2k_3\ldots}{k}} \ln\tan\left(\frac{\pi}{4} + \frac{\Phi}{2}\right)$$

where

**34.10** $$k_1 = \frac{2\sqrt{k}}{1 + k}, \quad k_2 = \frac{2\sqrt{k_1}}{1 + k_1}, \quad \ldots \quad \text{and} \quad \Phi = \lim_{n\to\infty} \phi_n$$

The result is used in the approximate evaluation of $F(k, \phi)$.

## JACOBI'S ELLIPTIC FUNCTIONS

From 34.1 we define the following elliptic functions.

**34.11** $$x = \sin(\text{am}\, u) = \text{sn}\, u$$

**34.12** $$\sqrt{1 - x^2} = \cos(\text{am}\, u) = \text{cn}\, u$$

**34.13** $$\sqrt{1 - k^2x^2} = \sqrt{1 - k^2 \,\text{sn}^2 u} = \text{dn}\, u$$

We can also define the inverse functions $\text{sn}^{-1} x$, $\text{cn}^{-1} x$, $\text{dn}^{-1} x$ and the following

**34.14** $\text{ns}\, u = \dfrac{1}{\text{sn}\, u}$

**34.15** $\text{nc}\, u = \dfrac{1}{\text{cn}\, u}$

**34.16** $\text{nd}\, u = \dfrac{1}{\text{dn}\, u}$

**34.17** $\text{sc}\, u = \dfrac{\text{sn}\, u}{\text{cn}\, u}$

**34.18** $\text{sd}\, u = \dfrac{\text{sn}\, u}{\text{dn}\, u}$

**34.19** $\text{cd}\, u = \dfrac{\text{cn}\, u}{\text{dn}\, u}$

**34.20** $\text{cs}\, u = \dfrac{\text{cn}\, u}{\text{sn}\, u}$

**34.21** $\text{dc}\, u = \dfrac{\text{dn}\, u}{\text{cn}\, u}$

**34.22** $\text{ds}\, u = \dfrac{\text{dn}\, u}{\text{sn}\, u}$

## ADDITION FORMULAS

**34.23** $$\text{sn}(u + v) = \frac{\text{sn}\, u \,\text{cn}\, v \,\text{dn}\, v + \text{cn}\, u \,\text{sn}\, v \,\text{dn}\, u}{1 - k^2 \,\text{sn}^2 u \,\text{sn}^2 v}$$

**34.24** $$\text{cn}(u + v) = \frac{\text{cn}\, u \,\text{cn}\, v - \text{sn}\, u \,\text{sn}\, v \,\text{dn}\, u \,\text{dn}\, v}{1 - k^2 \,\text{sn}^2 u \,\text{sn}^2 v}$$

**34.25** $$\text{dn}(u + v) = \frac{\text{dn}\, u \,\text{dn}\, v - k^2 \,\text{sn}\, u \,\text{sn}\, v \,\text{cn}\, u \,\text{cn}\, v}{1 - k^2 \,\text{sn}^2 u \,\text{sn}^2 v}$$

## DERIVATIVES

**34.26** $$\frac{d}{du}\operatorname{sn} u = \operatorname{cn} u \operatorname{dn} u$$

**34.27** $$\frac{d}{du}\operatorname{cn} u = -\operatorname{sn} u \operatorname{dn} u$$

**34.28** $$\frac{d}{du}\operatorname{dn} u = -k^2 \operatorname{sn} u \operatorname{cn} u$$

**34.29** $$\frac{d}{du}\operatorname{sc} u = \operatorname{dc} u \operatorname{nc} u$$

## SERIES EXPANSIONS

**34.30** $$\operatorname{sn} u = u - (1+k^2)\frac{u^3}{3!} + (1+14k^2+k^4)\frac{u^5}{5!} - (1+135k^2+135k^4+k^6)\frac{u^7}{7!} + \cdots$$

**34.31** $$\operatorname{cn} u = 1 - \frac{u^2}{2!} + (1+4k^2)\frac{u^4}{4!} - (1+44k^2+16k^4)\frac{u^6}{6!} + \cdots$$

**34.32** $$\operatorname{dn} u = 1 - k^2\frac{u^2}{2!} + k^2(4+k^2)\frac{u^4}{4!} - k^2(16+44k^2+k^4)\frac{u^6}{6!} + \cdots$$

## CATALAN'S CONSTANT

**34.33** $$\frac{1}{2}\int_0^1 K\,dk = \frac{1}{2}\int_{k=0}^1\int_{\theta=0}^{\pi/2}\frac{d\theta\,dk}{\sqrt{1-k^2\sin^2\theta}} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \cdots = .915965594\ldots$$ $\underline{\pi^2}$

## PERIODS OF ELLIPTIC FUNCTIONS

Let

**34.34** $$K = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \qquad K' = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k'^2\sin^2\theta}} \qquad \text{where } k' = \sqrt{1-k^2}$$

Then

**34.35** $\operatorname{sn} u$ has periods $4K$ and $2iK'$

**34.36** $\operatorname{cn} u$ has periods $4K$ and $2K + 2iK'$

**34.37** $\operatorname{dn} u$ has periods $2K$ and $4iK'$

## IDENTITIES INVOLVING ELLIPTIC FUNCTIONS

**34.38** $$\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1$$

**34.39** $$\operatorname{dn}^2 u + k^2\operatorname{sn}^2 u = 1$$

**34.40** $$\operatorname{dn}^2 u - k^2\operatorname{cn}^2 u = k'^2 \quad \text{where } k' = \sqrt{1-k^2}$$

**34.41** $$\operatorname{sn}^2 u = \frac{1-\operatorname{cn} 2u}{1+\operatorname{dn} 2u}$$

**34.42** $$\operatorname{cn}^2 u = \frac{\operatorname{dn} 2u + \operatorname{cn} 2u}{1+\operatorname{dn} 2u}$$

**34.43** $$\operatorname{dn}^2 u = \frac{1-k^2+\operatorname{dn} 2u + k^2\operatorname{cn} u}{1+\operatorname{dn} 2u}$$

**34.44** $$\sqrt{\frac{1-\operatorname{cn} 2u}{1+\operatorname{cn} 2u}} = \frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u}$$

**34.45** $$\sqrt{\frac{1-\operatorname{dn} 2u}{1+\operatorname{dn} 2u}} = \frac{k\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u}$$

## SPECIAL VALUES

**34.46** $\text{sn}\,0 = 0$ **34.47** $\text{cn}\,0 = 1$ **34.48** $\text{dn}\,0 = 1$ **34.49** $\text{sc}\,0 = 0$ **34.50** $\text{am}\,0 = 0$

## INTEGRALS

**34.51** $\int \text{sn}\,u\,du = \frac{1}{k}\ln(\text{dn}\,u - k\,\text{cn}\,u)$

**34.52** $\int \text{cn}\,u\,du = \frac{1}{k}\cos^{-1}(\text{dn}\,u)$

**34.53** $\int \text{dn}\,u\,du = \sin^{-1}(\text{sn}\,u)$

**34.54** $\int \text{sc}\,u\,du = \frac{1}{\sqrt{1-k^2}}\ln(\text{dc}\,u + \sqrt{1-k^2}\,\text{nc}\,u)$

**34.55** $\int \text{cs}\,u\,du = \ln(\text{ns}\,u - \text{ds}\,u)$

**34.56** $\int \text{cd}\,u\,du = \frac{1}{k}\ln(\text{nd}\,u + k\,\text{sd}\,u)$

**34.57** $\int \text{dc}\,u\,du = \ln(\text{nc}\,u + \text{sc}\,u)$

**34.58** $\int \text{sd}\,u\,du = \frac{-1}{k\sqrt{1-k^2}}\sin^{-1}(k\,\text{cd}\,u)$

**34.59** $\int \text{ds}\,u\,du = \ln(\text{ns}\,u - \text{cs}\,u)$

**34.60** $\int \text{ns}\,u\,du = \ln(\text{ds}\,u - \text{cs}\,u)$

**34.61** $\int \text{nc}\,u\,du = \frac{1}{\sqrt{1-k^2}}\ln\left(\text{dc}\,u + \frac{\text{sc}\,u}{\sqrt{1-k^2}}\right)$

**34.62** $\int \text{nd}\,u\,du = \frac{1}{\sqrt{1-k^2}}\cos^{-1}(\text{cd}\,u)$

## LEGENDRE'S RELATION

**34.63** $$EK' + E'K - KK' = \pi/2$$

where

**34.64** $$E = \int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}\,d\theta \qquad K = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

**34.65** $$E' = \int_0^{\pi/2}\sqrt{1-k'^2\sin^2\theta} \qquad K' = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k'^2\sin^2\theta}}$$

# 35 MISCELLANEOUS SPECIAL FUNCTIONS

## ERROR FUNCTION $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du$

**35.1** $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots\right)$

**35.2** $\operatorname{erf}(x) \sim 1 - \frac{e^{-x^2}}{\sqrt{\pi}\,x}\left(1 - \frac{1}{2x^2} + \frac{1\cdot 3}{(2x^2)^2} - \frac{1\cdot 3\cdot 5}{(2x^2)^3} + \cdots\right)$

**35.3** $\operatorname{erf}(-x) = -\operatorname{erf}(x), \quad \operatorname{erf}(0) = 0, \quad \operatorname{erf}(\infty) = 1$

## COMPLEMENTARY ERROR FUNCTION $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}\,du$

**35.4** $\operatorname{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots\right)$

**35.5** $\operatorname{erfc}(x) \sim \frac{e^{-x^2}}{\sqrt{\pi}\,x}\left(1 - \frac{1}{2x^2} + \frac{1\cdot 3}{(2x^2)^2} - \frac{1\cdot 3\cdot 5}{(2x^2)^3} + \cdots\right)$

**35.6** $\operatorname{erfc}(0) = 1, \quad \operatorname{erfc}(\infty) = 0$

## EXPONENTIAL INTEGRAL $Ei(x) = \int_x^\infty \frac{e^{-u}}{u}\,du$

**35.7** $Ei(x) = -\gamma - \ln x + \int_0^x \frac{1 - e^{-u}}{u}\,du$

**35.8** $Ei(x) = -\gamma - \ln x + \left(\frac{x}{1\cdot 1!} - \frac{x^2}{2\cdot 2!} + \frac{x^3}{3\cdot 3!} - \cdots\right)$

**35.9** $Ei(x) \sim \frac{e^{-x}}{x}\left(1 - \frac{1!}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} + \cdots\right)$

**35.10** $Ei(\infty) = 0$

## SINE INTEGRAL $Si(x) = \int_0^x \frac{\sin u}{u}\,du$

**35.11** $Si(x) = \frac{x}{1\cdot 1!} - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \cdots$

**35.12** $Si(x) \sim \frac{\pi}{2} - \frac{\sin x}{x}\left(\frac{1}{x} - \frac{3!}{x^3} + \frac{5!}{x^5} - \cdots\right) - \frac{\cos x}{x}\left(1 - \frac{2!}{x^2} + \frac{4!}{x^4} - \cdots\right)$

**35.13** $Si(-x) = -Si(x), \quad Si(0) = 0, \quad Si(\infty) = \pi/2$

## COSINE INTEGRAL $Ci(x) = \int_x^\infty \frac{\cos u}{u}\,du$

**35.14** $$Ci(x) = -\gamma - \ln x + \int_0^x \frac{1-\cos u}{u}\,du$$

**35.15** $$Ci(x) = -\gamma - \ln x + \frac{x^2}{2\cdot 2!} - \frac{x^4}{4\cdot 4!} + \frac{x^6}{6\cdot 6!} - \frac{x^8}{8\cdot 8!} + \cdots$$

**35.16** $$Ci(x) \sim \frac{\cos x}{x}\left(\frac{1}{x} - \frac{3!}{x^3} + \frac{5!}{x^5} - \cdots\right) - \frac{\sin x}{x}\left(1 - \frac{2!}{x^2} + \frac{4!}{x^4} - \cdots\right)$$

**35.17** $$Ci(\infty) = 0$$

## FRESNEL SINE INTEGRAL $S(x) = \sqrt{\frac{2}{\pi}}\int_0^x \sin u^2\,du$

**35.18** $$S(x) = \sqrt{\frac{2}{\pi}}\left(\frac{x^3}{3\cdot 1!} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \cdots\right)$$

**35.19** $$S(x) \sim \frac{1}{2} - \frac{1}{\sqrt{2\pi}}\left\{(\cos x^2)\left(\frac{1}{x} - \frac{1\cdot 3}{2^2x^5} + \frac{1\cdot 3\cdot 5\cdot 7}{2^4x^9} - \cdots\right) + (\sin x^2)\left(\frac{1}{2x^3} - \frac{1\cdot 3\cdot 5}{2^3x^7} + \cdots\right)\right\}$$

**35.20** $$S(-x) = -S(x), \quad S(0) = 0, \quad S(\infty) = \frac{1}{2}$$

## FRESNEL COSINE INTEGRAL $C(x) = \sqrt{\frac{2}{\pi}}\int_0^x \cos u^2\,du$

**35.21** $$C(x) = \sqrt{\frac{2}{\pi}}\left(\frac{x}{1!} - \frac{x^5}{5\cdot 2!} + \frac{x^9}{9\cdot 4!} - \frac{x^{13}}{13\cdot 6!} + \cdots\right)$$

**35.22** $$C(x) \sim \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\left\{(\sin x^2)\left(\frac{1}{x} - \frac{1\cdot 3}{2^2x^5} + \frac{1\cdot 3\cdot 5\cdot 7}{2^4x^9} - \cdots\right) - (\cos x^2)\left(\frac{1}{2x^3} - \frac{1\cdot 3\cdot 5}{2^3x^7} + \cdots\right)\right\}$$

**35.23** $$C(-x) = -C(x), \quad C(0) = 0, \quad C(\infty) = \frac{1}{2}$$

## RIEMANN ZETA FUNCTION $\zeta(x) = \frac{1}{1^x} + \frac{1}{2^x} + \frac{1}{3^x} + \cdots$

**35.24** $$\zeta(x) = \frac{1}{\Gamma(x)}\int_0^\infty \frac{u^{x-1}}{e^u - 1}\,du, \qquad x > 1$$

**35.25** $$\zeta(1-x) = 2^{1-x}\pi^{-x}\,\Gamma(x)\cos(\pi x/2)\,\zeta(x) \qquad \text{[extension to other values]}$$

**35.26** $$\zeta(2k) = \frac{2^{2k-1}\pi^{2k}B_k}{(2k)!} \qquad k = 1, 2, 3, \ldots$$

# 36 INEQUALITIES

## TRIANGLE INEQUALITY

**36.1**
$$|a_1| - |a_2| \leqq |a_1 + a_2| \leqq |a_1| + |a_2|$$

**36.2**
$$|a_1 + a_2 + \cdots + a_n| \leqq |a_1| + |a_2| + \cdots + |a_n|$$

## CAUCHY-SCHWARZ INEQUALITY

**36.3**
$$|a_1b_1 + a_2b_2 + \cdots + a_nb_n|^2 \leqq (|a_1|^2 + |a_2|^2 + \cdots + |a_n|^2)(|b_1|^2 + |b_2|^2 + \cdots + |b_n|^2)$$

The equality holds if and only if $a_1/b_1 = a_2/b_2 = \cdots = a_n/b_n$.

## INEQUALITIES INVOLVING ARITHMETIC, GEOMETRIC AND HARMONIC MEANS

If $A$, $G$ and $H$ are the arithmetic, geometric and harmonic means of the positive numbers $a_1, a_2, \ldots, a_n$, then

**36.4**
$$H \leqq G \leqq A$$

where

**36.5** $A = \dfrac{a_1 + a_2 + \cdots + a_n}{n}$ **36.6** $G = \sqrt[n]{a_1a_2\ldots a_n}$ **36.7** $\dfrac{1}{H} = \dfrac{1}{n}\left(\dfrac{1}{a_1} + \dfrac{1}{a_2} + \cdots + \dfrac{1}{a_n}\right)$

The equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

## HOLDER'S INEQUALITY

**36.8**
$$|a_1b_1 + a_2b_2 + \cdots + a_nb_n| \leqq (|a_1|^p + |a_2|^p + \cdots + |a_n|^p)^{1/p}(|b_1|^q + |b_2|^q + \cdots + |b_n|^q)^{1/q}$$

where

**36.9**
$$\frac{1}{p} + \frac{1}{q} = 1 \qquad p > 1,\ q > 1$$

The equality holds if and only if $|a_1|^{p-1}/|b_1| = |a_2|^{p-1}/|b_2| = \cdots = |a_n|^{p-1}/|b_n|$. For $p = q = 2$ it reduces to 36.3.

## CHEBYSHEV'S INEQUALITY

If $a_1 \geqq a_2 \geqq \cdots \geqq a_n$ and $b_1 \geqq b_2 \geqq \cdots \geqq b_n$, then

**36.10** $$\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)\left(\frac{b_1 + b_2 + \cdots + b_n}{n}\right) \leqq \frac{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}{n}$$

or

**36.11** $$(a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n) \leqq n(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)$$

## MINKOWSKI'S INEQUALITY

If $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots b_n$ are all positive and $p > 1$, then

**36.12** $$\{(a_1 + b_1)^p + (a_2 + b_2)^p + \cdots + (a_n + b_n)^p\}^{1/p} \leqq (a_1^p + a_2^p + \cdots + a_n^p)^{1/p} + (b_1^p + b_2^p + \cdots + b_n^p)^{1/p}$$

The equality holds if and only if $a_1/b_1 = a_2/b_2 = \cdots = a_n/b_n$.

## CAUCHY-SCHWARZ INEQUALITY FOR INTEGRALS

**36.13** $$\left|\int_a^b f(x)\, g(x)\, dx\right|^2 \leqq \left\{\int_a^b |f(x)|^2\, dx\right\}\left\{\int_a^b |g(x)|^2\, dx\right\}$$

The equality holds if and only if $f(x)/g(x)$ is a constant.

## HOLDER'S INEQUALITY FOR INTEGRALS

**36.14** $$\int_a^b |f(x)\, g(x)|\, dx \leqq \left\{\int_a^b |f(x)|^p\, dx\right\}^{1/p} \left\{\int_a^b |g(x)|^q\, dx\right\}^{1/q}$$

where $1/p + 1/q = 1$, $p > 1$, $q > 1$. If $p = q = 2$, this reduces to 36.13.

The equality holds if and only if $|f(x)|^{p-1}/|g(x)|$ is a constant.

## MINKOWSKI'S INEQUALITY FOR INTEGRALS

If $p > 1$,

**36.15** $$\left\{\int_a^b |f(x) + g(x)|^p\, dx\right\}^{1/p} \leqq \left\{\int_a^b |f(x)|^p\, dx\right\}^{1/p} + \left\{\int_a^b |g(x)|^p\, dx\right\}^{1/p}$$

The equality holds if and only if $f(x)/g(x)$ is a constant.

# 37 PARTIAL FRACTION EXPANSIONS

**37.1** $$\cot x = \frac{1}{x} + 2x\left\{\frac{1}{x^2-\pi^2} + \frac{1}{x^2-4\pi^2} + \frac{1}{x^2-9\pi^2} + \cdots\right\}$$

**37.2** $$\csc x = \frac{1}{x} - 2x\left\{\frac{1}{x^2-\pi^2} - \frac{1}{x^2-4\pi^2} + \frac{1}{x^2-9\pi^2} - \cdots\right\}$$

**37.3** $$\sec x = 4\pi\left\{\frac{1}{\pi^2-4x^2} - \frac{3}{9\pi^2-4x^2} + \frac{5}{25\pi^2-4x^2} - \cdots\right\}$$

**37.4** $$\tan x = 8x\left\{\frac{1}{\pi^2-4x^2} + \frac{1}{9\pi^2-4x^2} + \frac{1}{25\pi^2-4x^2} + \cdots\right\}$$

**37.5** $$\sec^2 x = 4\left\{\frac{1}{(\pi-2x)^2} + \frac{1}{(\pi+2x)^2} + \frac{1}{(3\pi-2x)^2} + \frac{1}{(3\pi+2x)^2} + \cdots\right\}$$

**37.6** $$\csc^2 x = \frac{1}{x^2} + \frac{1}{(x-\pi)^2} + \frac{1}{(x+\pi)^2} + \frac{1}{(x-2\pi)^2} + \frac{1}{(x+2\pi)^2} + \cdots$$

**37.7** $$\coth x = \frac{1}{x} + 2x\left\{\frac{1}{x^2+\pi^2} + \frac{1}{x^2+4\pi^2} + \frac{1}{x^2+9\pi^2} + \cdots\right\}$$

**37.8** $$\operatorname{csch} x = \frac{1}{x} - 2x\left\{\frac{1}{x^2+\pi^2} - \frac{1}{x^2+4\pi^2} + \frac{1}{x^2+9\pi^2} - \cdots\right\}$$

**37.9** $$\operatorname{sech} x = 4\pi\left\{\frac{1}{\pi^2+4x^2} - \frac{3}{9\pi^2+4x^2} + \frac{5}{25\pi^2+4x^2} - \cdots\right\}$$

**37.10** $$\tanh x = 8x\left\{\frac{1}{\pi^2+4x^2} + \frac{1}{9\pi^2+4x^2} + \frac{1}{25\pi^2+4x^2} + \cdots\right\}$$

# 38 INFINITE PRODUCTS

**38.1** $$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots$$

**38.2** $$\cos x = \left(1-\frac{4x^2}{\pi^2}\right)\left(1-\frac{4x^2}{9\pi^2}\right)\left(1-\frac{4x^2}{25\pi^2}\right)\cdots$$

**38.3** $$\sinh x = x\left(1+\frac{x^2}{\pi^2}\right)\left(1+\frac{x^2}{4\pi^2}\right)\left(1+\frac{x^2}{9\pi^2}\right)\cdots$$

**38.4** $$\cosh x = \left(1+\frac{4x^2}{\pi^2}\right)\left(1+\frac{4x^2}{9\pi^2}\right)\left(1+\frac{4x^2}{25\pi^2}\right)\cdots$$

**38.5** $$\frac{1}{\Gamma(x)} = xe^{\gamma x}\left\{\left(1+\frac{x}{1}\right)e^{-x}\right\}\left\{\left(1+\frac{x}{2}\right)e^{-x/2}\right\}\left\{\left(1+\frac{x}{3}\right)e^{-x/3}\right\}\cdots$$

See also 16.12, page 102.

**38.6** $$J_0(x) = \left(1-\frac{x^2}{\lambda_1^2}\right)\left(1-\frac{x^2}{\lambda_2^2}\right)\left(1-\frac{x^2}{\lambda_3^2}\right)\cdots$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive roots of $J_0(x) = 0$.

**38.7** $$J_1(x) = x\left(1-\frac{x^2}{\lambda_1^2}\right)\left(1-\frac{x^2}{\lambda_2^2}\right)\left(1-\frac{x^2}{\lambda_3^2}\right)\cdots$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive roots of $J_1(x) = 0$.

**38.8** $$\frac{\sin x}{x} = \cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}\cos\frac{x}{16}\cdots$$

**38.9** $$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots$$

This is called *Wallis' product.*

# 39 PROBABILITY DISTRIBUTIONS

## BINOMIAL DISTRIBUTION

**39.1** $$\Phi(x) = \sum_{t \leqq x} \binom{n}{t} p^t q^{n-t} \qquad p > 0,\ q > 0,\ p + q = 1$$

## POISSON DISTRIBUTION

**39.2** $$\Phi(x) = \sum_{t \leqq x} \frac{\lambda^t e^{-\lambda}}{t!} \qquad \lambda > 0$$

## HYPERGEOMETRIC DISTRIBUTION

**39.3** $$\Phi(x) = \sum_{t \leqq x} \frac{\binom{r}{t}\binom{s}{n-t}}{\binom{r+s}{n}}$$

## NORMAL DISTRIBUTION

**39.4** $$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2}\, dt$$

## STUDENT'S $t$ DISTRIBUTION

**39.5** $$\Phi(x) = \frac{1}{\sqrt{n\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(n/2)} \int_{-\infty}^{x} \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2} dt$$

## CHI SQUARE DISTRIBUTION

**39.6** $$\Phi(x) = \frac{1}{2^{n/2}\,\Gamma(n/2)} \int_{0}^{x} t^{(n-2)/2} e^{-t/2}\, dt$$

## $F$ DISTRIBUTION

**39.7** $$\Phi(x) = \frac{\Gamma\left(\frac{n_1+n_2}{2}\right) n_1^{n_1/2} n_2^{n_2/2}}{\Gamma(n_1/2)\,\Gamma(n_2/2)} \int_{0}^{x} t^{n_1/2}(n_2 + n_1 t)^{-(n_1+n_2)/2}\, dt$$

# 40 SPECIAL MOMENTS OF INERTIA

The table below shows the moments of inertia of various rigid bodies of mass $M$. In all cases it is assumed the body has uniform [i.e. constant] density.

| TYPE OF RIGID BODY | MOMENT OF INERTIA |
|---|---|
| **40.1** Thin rod of length $a$ | |
| (*a*) about axis perpendicular to the rod through the center of mass, | $\frac{1}{12}Ma^2$ |
| (*b*) about axis perpendicular to the rod through one end. | $\frac{1}{3}Ma^2$ |
| **40.2** Rectangular parallelepiped with sides $a, b, c$ | |
| (*a*) about axis parallel to $c$ and through center of face $ab$, | $\frac{1}{12}M(a^2+b^2)$ |
| (*b*) about axis through center of face $bc$ and parallel to $c$. | $\frac{1}{12}M(4a^2+b^2)$ |
| **40.3** Thin rectangular plate with sides $a, b$ | |
| (*a*) about axis perpendicular to the plate through center, | $\frac{1}{12}M(a^2+b^2)$ |
| (*b*) about axis parallel to side $b$ through center. | $\frac{1}{12}Ma^2$ |
| **40.4** Circular cylinder of radius $a$ and height $h$ | |
| (*a*) about axis of cylinder, | $\frac{1}{2}Ma^2$ |
| (*b*) about axis through center of mass and perpendicular to cylindrical axis, | $\frac{1}{12}M(h^2+3a^2)$ |
| (*c*) about axis coinciding with diameter at one end. | $\frac{1}{12}M(4h^2+3a^2)$ |
| **40.5** Hollow circular cylinder of outer radius $a$, inner radius $b$ and height $h$ | |
| (*a*) about axis of cylinder, | $\frac{1}{2}M(a^2+b^2)$ |
| (*b*) about axis through center of mass and perpendicular to cylindrical axis, | $\frac{1}{12}M(3a^2+3b^2+h^2)$ |
| (*c*) about axis coinciding with diameter at one end. | $\frac{1}{12}M(3a^2+3b^2+4h^2)$ |

| | |
|---|---|
| **40.6** Circular plate of radius $a$ | |
| (a) about axis perpendicular to plate through center,<br>(b) about axis coinciding with a diameter. | $\frac{1}{2}Ma^2$<br>$\frac{1}{4}Ma^2$ |
| **40.7** Hollow circular plate or ring with outer radius $a$ and inner radius $b$ | |
| (a) about axis perpendicular to plane of plate through center,<br>(b) about axis coinciding with a diameter. | $\frac{1}{2}M(a^2+b^2)$<br>$\frac{1}{4}M(a^2+b^2)$ |
| **40.8** Thin circular ring of radius $a$ | |
| (a) about axis perpendicular to plane of ring through center,<br>(b) about axis coinciding with diameter. | $Ma^2$<br>$\frac{1}{2}Ma^2$ |
| **40.9** Sphere of radius $a$ | |
| (a) about axis coinciding with a diameter,<br>(b) about axis tangent to the surface. | $\frac{2}{5}Ma^2$<br>$\frac{7}{5}Ma^2$ |
| **40.10** Hollow sphere of outer radius $a$ and inner radius $b$ | |
| (a) about axis coinciding with a diameter,<br>(b) about axis tangent to the surface. | $\frac{2}{5}M(a^5-b^5)/(a^3-b^3)$<br>$\frac{2}{5}M(a^5-b^5)/(a^3-b^3) + Ma^2$ |
| **40.11** Hollow spherical shell of radius $a$ | |
| (a) about axis coinciding with a diameter,<br>(b) about axis tangent to the surface. | $Ma^2$<br>$2Ma^2$ |
| **40.12** Ellipsoid with semi-axes $a, b, c$ | |
| (a) about axis coinciding with semi-axis $c$,<br>(b) about axis tangent to surface, parallel to semi-axis $c$ and at distance $a$ from center. | $\frac{1}{5}M(a^2+b^2)$<br>$\frac{1}{5}M(6a^2+b^2)$ |
| **40.13** Circular cone of radius $a$ and height $h$ | |
| (a) about axis of cone,<br>(b) about axis through vertex and perpendicular to axis,<br>(c) about axis through center of mass and perpendicular to axis. | $\frac{3}{10}Ma^2$<br>$\frac{3}{20}M(a^2+4h^2)$<br>$\frac{3}{80}M(4a^2+h^2)$ |
| **40.14** Torus with outer radius $a$ and inner radius $b$ | |
| (a) about axis through center of mass and perpendicular to plane of torus,<br>(b) about axis through center of mass and in the plane of the torus. | $\frac{1}{4}M(7a^2-6ab+3b^2)$<br>$\frac{1}{4}M(9a^2-10ab+5b^2)$ |

# 41

# CONVERSION FACTORS

**Length**

1 kilometer (km) = 1000 meters (m)
1 meter (m) = 100 centimeters (cm)
1 centimeter (cm) = $10^{-2}$ m
1 millimeter (mm) = $10^{-3}$ m
1 micron ($\mu$) = $10^{-6}$ m
1 millimicron (m$\mu$) = $10^{-9}$ m
1 angstrom (A) = $10^{-10}$ m

1 inch (in.) = 2.540 cm
1 foot (ft) = 30.48 cm
1 mile (mi) = 1.609 km
1 mil = $10^{-3}$ in.
1 centimeter = 0.3937 in.
1 meter = 39.37 in.
1 kilometer = 0.6214 mile

**Area**

1 square meter (m$^2$) = 10.76 ft$^2$
1 square foot (ft$^2$) = 929 cm$^2$

1 square mile (mi$^2$) = 640 acres
1 acre = 43,560 ft$^2$

**Volume**

1 liter ($l$) = 1000 cm$^3$ = 1.057 quart (qt) = 61.02 in$^3$ = 0.03532 ft$^3$
1 cubic meter (m$^3$) = 1000 $l$ = 35.32 ft$^3$
1 cubic foot (ft$^3$) = 7.481 U.S. gal = 0.02832 m$^3$ = 28.32 $l$
1 U.S. gallon (gal) = 231 in$^3$ = 3.785 $l$; 1 British gallon = 1.201 U.S. gallon = 277.4 in$^3$

**Mass**

1 kilogram (kg) = 2.2046 pounds (lb) = 0.06852 slug; 1 lb = 453.6 gm = 0.03108 slug
1 slug = 32.174 lb = 14.59 kg

**Speed**

1 km/hr = 0.2778 m/sec = 0.6214 mi/hr = 0.9113 ft/sec
1 mi/hr = 1.467 ft/sec = 1.609 km/hr = 0.4470 m/sec

**Density**

1 gm/cm$^3$ = $10^3$ kg/m$^3$ = 62.43 lb/ft$^3$ = 1.940 slug/ft$^3$
1 lb/ft$^3$ = 0.01602 gm/cm$^3$; 1 slug/ft$^3$ = 0.5154 gm/cm$^3$

**Force**

1 newton (nt) = $10^5$ dynes = 0.1020 kgwt = 0.2248 lbwt
1 pound weight (lbwt) = 4.448 nt = 0.4536 kgwt = 32.17 poundals
1 kilogram weight (kgwt) = 2.205 lbwt = 9.807 nt
1 U.S. short ton = 2000 lbwt; 1 long ton = 2240 lbwt; 1 metric ton = 2205 lbwt

**Energy**

1 joule = 1 nt m = $10^7$ ergs = 0.7376 ft lbwt = 0.2389 cal = $9.481 \times 10^{-4}$ Btu
1 ft lbwt = 1.356 joules = 0.3239 cal = $1.285 \times 10^{-3}$ Btu
1 calorie (cal) = 4.186 joules = 3.087 ft lbwt = $3.968 \times 10^{-3}$ Btu
1 Btu (British thermal unit) = 778 ft lbwt = 1055 joules = 0.293 watt hr
1 kilowatt hour (kw hr) = $3.60 \times 10^6$ joules = 860.0 kcal = 3413 Btu
1 electron volt (ev) = $1.602 \times 10^{-19}$ joule

**Power**

1 watt = 1 joule/sec = $10^7$ ergs/sec = 0.2389 cal/sec
1 horsepower (hp) = 550 ft lbwt/sec = 33,000 ft lbwt/min = 745.7 watts
1 kilowatt (kw) = 1.341 hp = 737.6 ft lbwt/sec = 0.9483 Btu/sec

**Pressure**

1 nt/m$^2$ = 10 dynes/cm$^2$ = $9.869 \times 10^{-6}$ atmosphere = $2.089 \times 10^{-2}$ lbwt/ft$^2$
1 lbwt/in$^2$ = 6895 nt/m$^2$ = 5.171 cm mercury = 27.68 in. water
1 atmosphere (atm) = $1.013 \times 10^5$ nt/m$^2$ = $1.013 \times 10^6$ dynes/cm$^2$ = 14.70 lbwt/in$^2$
= 76 cm mercury = 406.8 in. water

## *Part II*

# TABLES

# SAMPLE PROBLEMS

## ILLUSTRATING USE OF THE TABLES

### COMMON LOGARITHMS

1. Find log 2.36.

We must find the number $p$ such that $10^p = 2.36 = N$. Since $10^0 = 1$ and $10^1 = 10$, $p$ lies between 0 and 1 and can be found from the tables of common logarithms on page 202.

Thus to find log 2.36 we glance down the *left* column headed $N$ until we come to the first two digits, 23. Then we proceed *right* to the column headed 6. We find the entry 3729. Thus $\log 2.36 = 0.3729$, i.e. $2.36 = 10^{0.3729}$.

2. Find (*a*) log 23.6, (*b*) log 236, (*c*) log 2360.

From Problem 1, $2.36 = 10^{0.3729}$. Then multiplying successively by 10 we have

$$23.6 = 10^{1.3729}, \quad 236 = 10^{2.3729}, \quad 2360 = 10^{3.3729}$$

Thus

(*a*) $\log 23.6 = 1.3729$

(*b*) $\log 236 = 2.3729$

(*c*) $\log 2360 = 3.3729$.

The number .3729 obtained from the table is called the *mantissa* of the logarithm. The number before the decimal point is called the *characteristic*. Thus in (*b*) the characteristic is 2.

The following rule is easily demonstrated.

**Rule 1.** For a number greater than 1, the characteristic is one less than the number of digits before the decimal point. For example since 2360 has four digits before the decimal point, the characteristic is $4 - 1 = 3$.

3. Find (*a*) log .236, (*b*) log .0236, (*c*) log .00236.

From Problem 1, $2.36 = 10^{0.3729}$. Then dividing successively by 10 we have

$$.236 = 10^{0.3729-1} = 10^{9.3729-10} = 10^{-.6271}$$

$$.0236 = 10^{0.3729-2} = 10^{8.3729-10} = 10^{-1.6271}$$

$$.00236 = 10^{0.3729-3} = 10^{7.3729-10} = 10^{-2.6271}$$

Then

(*a*) $\log .236 = 9.3729 - 10 = -.6271$

(*b*) $\log .0236 = 8.3729 - 10 = -1.6271$

(*c*) $\log .00236 = 7.3729 - 10 = -2.6271$.

The number .3729 is the mantissa of the logarithm. The number apart from the mantissa [for example $9 - 10$, $8 - 10$ or $7 - 10$] is the characteristic.

The following rule is easily demonstrated.

**Rule 2.** For a positive number less than 1, the characteristic is negative and numerically one more than the number of zeros immediately following the decimal point. For example since .00236 has two zeros immediately following the decimal point, the characteristic is $-3$ or $7 - 10$.

**4.** Verify each of the following logarithms.

(*a*) log 87.2. Mantissa = .9405, characteristic = 1; then log 87.2 = 1.9405.

(*b*) log 395,000 = 5.5966.

(*c*) log .0482. Mantissa = .6830, characteristic = 8 − 10; then $\log_{10} .0482 = 8.6830 - 10$.

(*d*) log .000827 = 6.9175 − 10.

**5.** Find log 4.638.

Since the number has four digits, we must use interpolation to find the mantissa. The mantissa of log 4638 is .8 of the way between the mantissas of log 4630 and log 4640.

$$\begin{aligned} \text{Mantissa of } \log 4640 &= .6665 \\ \text{Mantissa of } \log 4630 &= \underline{.6656} \\ \text{Tabular difference} &= .0009 \end{aligned} \qquad \begin{aligned} \text{Mantissa of } \log 4.638 &= .6656 + (.8)(.0009) \\ &= .6663 \text{ to four digits} \\ \text{Then } \log 4.638 &= 0.6663 \end{aligned}$$

If desired the proportional parts table on page 202 can be used to give the mantissa directly (6656 + 7).

**6.** Verify each of the following logarithms.

(*a*) log 183.2 = 2.2630 (2625 + 5)

(*b*) log 87,640 = 4.9427 (9425 + 2)

(*c*) log .2548 = 9.4062 − 10 (4048 + 14)

(*d*) log .009848 = 7.9933 − 10 (9930 + 3)

## COMMON ANTILOGARITHMS

**7.** Find (*a*) antilog 1.7530, (*b*) antilog (7.7530 − 10).

(*a*) We must find the value of $10^{1.7530}$. Since the mantissa is .7530 we glance down the *left* column headed $p$ in the table on page 205 until we come to the first two digits 75. Then we proceed *right* to the column headed 3. We find the entry 5662. Since the characteristic is 1, there are two digits before the decimal point. Then the required number is 56.62.

(*b*) As in part (*a*) we find the entry 5662 corresponding to the mantissa .7530. Then since the characteristic is 7 − 10, the number must have two zeros immediately following the decimal point. Thus the required number is .005662.

**8.** Find antilog (9.3842 − 10).

The mantissa .3842 lies between .3840 and .3850 and we must use interpolation. From the table on page 204 we have

$$\begin{aligned} \text{Number corresponding to } .3850 &= 2427 \\ \text{Number corresponding to } .3840 &= \underline{2421} \\ \text{Tabular difference} &= \quad 6 \end{aligned} \qquad \begin{aligned} \text{Given mantissa} &= .3842 \\ \text{Next smaller mantissa} &= \underline{.3840} \\ \text{Difference} &= .0002 \end{aligned}$$

Then $2421 + \frac{2}{10}(2427 - 2421) = 2422$ to four digits, and the required number is 0.2422.

The proportional parts table on page 204 can also be used.

**9.** Verify each of the following antilogarithm.

(*a*) antilog 2.6715 = 469.3

(*b*) antilog 9.6089 − 10 = .4063

(*c*) antilog 4.2023 = 15,930

## COMPUTATIONS USING LOGARITHMS

**10.** $P = \dfrac{(784.6)(.0431)}{28.23}$. $\log P = \log 784.6 + \log .0431 - \log 28.23$.

$$\begin{aligned} \log 784.6 &= 2.8947 \\ (+)\ \log .0431 &= \underline{8.6345 - 10} \\ &\phantom{=}\ 11.5292 - 10 \\ (-)\ \log 28.23 &= \underline{1.4507} \\ \log P &= 10.0785 - 10 = .0785. \quad \text{Then } P = 1.198. \end{aligned}$$

Note the exponential significance of the computation, i.e.

$$\frac{(784.6)(.0431)}{28.23} = \frac{(10^{2.8947})(10^{8.6345-10})}{10^{1.4507}} = 10^{2.8947+8.6345-10-1.4507} = 10^{.0785} = 1.198$$

**11.** $P = (5.395)^8$. $\log P = 8 \log 5.395 = 8(0.7320) = 5.8560$, and $P = 717{,}800$.

**12.** $P = \sqrt{387.2} = (387.2)^{1/2}$. $\log P = \frac{1}{2}\log 387.2 = \frac{1}{2}(2.5879) = 1.2940$ and $P = 19.68$.

**13.** $P = \sqrt[5]{.08317} = (.08317)^{1/5}$. $\log P = \frac{1}{5}\log .08317 = \frac{1}{5}(8.9200 - 10) = \frac{1}{5}(48.9200 - 50) = 9.7840 - 10$ and $P = .6081$.

**14.** $P = \dfrac{\sqrt{.003654}\,(18.37)^3}{(8.724)^4\,\sqrt[4]{743.8}}$. $\log P = \frac{1}{2}\log .003654 + 3\log 18.37 - (4\log 8.724 + \frac{1}{4}\log 743.8)$

*Numerator N*

$$\begin{aligned} \tfrac{1}{2}\log .003654 &= \tfrac{1}{2}(7.5628 - 10) \\ &= \tfrac{1}{2}(17.5628 - 20) = 8.7814 - 10 \\ 3\log 18.37 &= 3(1.2641) \qquad\quad = \underline{3.7923} \\ \text{Add:} \qquad & \qquad \log N = 12.5737 - 10 \end{aligned}$$

*Denominator D*

$$\begin{aligned} 4\log 8.724 &= 4(0.9407) = 3.7628 \\ \tfrac{1}{4}\log 743.6 &= \tfrac{1}{4}(2.8714) = \underline{0.7178} \\ \text{Add:} \qquad \log D &= 4.4806 \end{aligned}$$

$$\begin{aligned} \log N &= 12.5737 - 10 \\ (-)\ \log D &= \underline{4.4806} \\ \log P &= 8.0931 - 10. \quad \text{Then } P = .01239 \end{aligned}$$

## NATURAL OR NAPIERIAN LOGARITHMS

**15.** Find (*a*) ln 7.236, (*b*) ln 836.2, (*c*) ln .002548.

(*a*) Use the table on page 225.

$$\begin{aligned} \ln 7.240 &= 1.97962 \\ \ln 7.230 &= \underline{1.97824} \\ \text{Tabular difference} &= .00138 \end{aligned}$$

Then

$$\ln 7.236 = 1.97824 + \tfrac{6}{10}(.00138) = 1.97907$$

In terms of exponentials this means that $e^{1.97907} = 7.236$.

(*b*) As in part (*a*) we find

$$\ln 8.362 = 2.12346 + \tfrac{2}{10}(2.12465 - 2.12346) = 2.12370$$

Then

$$\ln 836.2 = \ln(8.362 \times 10^2) = \log 8.362 + 2\ln 10 = 2.12370 + 4.60517 = 6.72887$$

In terms of exponentials this means that $e^{6.72887} = 836.2$.

(*c*) As in part (*a*) we find

$$\ln 2.548 = 0.93216 + \tfrac{8}{10}(0.93609 - 0.93216) = 0.93530$$

Then

$$\ln .002548 = \ln(2.548 \times 10^{-3}) = \ln 2.548 - 3\ln 10 = 0.93530 - 6.90776 = -5.97246$$

In terms of exponentials this means that $e^{-5.97246} = .002548$.

## TRIGONOMETRIC FUNCTIONS (DEGREES AND MINUTES)

**16.** Find (*a*) $\sin 74°23'$, (*b*) $\cos 35°42'$, (*c*) $\tan 82°56'$.

(*a*) Refer to the table on page 206.

$$\begin{aligned} \sin 74°30' &= .9636 \\ \sin 74°20' &= \underline{.9628} \\ \text{Tabular difference} &= .0008 \end{aligned}$$

Then $$\sin 74°23' = .9628 + \tfrac{3}{10}(.0008) = .9630$$

(*b*) Refer to the table on page 207.

$$\begin{aligned} \cos 35°40' &= .8124 \\ \cos 35°50' &= \underline{.8107} \\ \text{Tabular difference} &= .0017 \end{aligned}$$

Then $$\cos 35°42' = .8124 - \tfrac{2}{10}(.0017) = .8121$$

or $$\cos 35°42' = .8107 + \tfrac{8}{10}(.0017) = .8121$$

(*c*) Refer to the table on page 208.

$$\begin{aligned} \tan 82°60' = \tan 83°0' &= 8.1443 \\ \tan 82°50' &= \underline{7.9530} \\ \text{Tabular difference} &= .1913 \end{aligned}$$

Then $$\tan 82°56' = 7.9530 + \tfrac{6}{10}(.1913) = 8.0678$$

**17.** Find (*a*) $\cot 45°16'$, (*b*) $\sec 73°48'$, (*c*) $\csc 28°33'$.

(*a*) Refer to the table on page 209.

$$\begin{aligned} \cot 45°10' &= .9942 \\ \cot 45°20' &= \underline{.9884} \\ \text{Tabular difference} &= .0058 \end{aligned}$$

Then $$\cot 45°16' = .9942 - \tfrac{6}{10}(.0058) = .9907$$

or $$\cot 45°16' = .9884 + \tfrac{4}{10}(.0058) = .9907$$

(*b*) Refer to the table on page 210.

$$\begin{aligned} \sec 73°50' &= 3.592 \\ \sec 73°40' &= \underline{3.556} \\ \text{Tabular difference} &= .036 \end{aligned}$$

Then $$\sec 73°48' = 3.556 + \tfrac{8}{10}(.036) = 3.585$$

(*c*) Refer to the table on page 211.

$$\begin{aligned} \csc 28°30' &= 2.096 \\ \csc 28°40' &= \underline{2.085} \\ \text{Tabular difference} &= .011 \end{aligned}$$

Then $$\csc 28°33' = 2.096 - \tfrac{3}{10}(.011) = 2.093$$

or $$\csc 28°33' = 2.085 + \tfrac{7}{10}(.011) = 2.093$$

## INVERSE TRIGONOMETRIC FUNCTIONS (DEGREES AND MINUTES)

18. Find (*a*) $\sin^{-1}(.2143)$, (*b*) $\cos^{-1}(.5412)$, (*c*) $\tan^{-1}(1.1536)$.

(*a*) Refer to the table on page 206.

$$\begin{aligned} \sin 12^\circ 30' &= .2164 \\ \sin 12^\circ 20' &= \underline{.2136} \\ \text{Tabular difference} &= .0028 \end{aligned}$$

Since .2143 is $\dfrac{.2143 - .2136}{.0028} = \frac{1}{4}$ of the way between .2136 and .2164, the required angle is $12^\circ 20' + \frac{1}{4}(10') = 12^\circ 22.5'$.

(*b*) Refer to the table on page 207.

$$\begin{aligned} \cos 57^\circ 10' &= .5422 \\ \cos 57^\circ 20' &= \underline{.5398} \\ \text{Tabular difference} &= .0024 \end{aligned}$$

Then $$\cos^{-1}(.5412) = 57^\circ 20' - \frac{.5412 - .5398}{.0024}(10') = 57^\circ 14.2'$$

or $$\cos^{-1}(.5412) = 57^\circ 10' + \frac{.5422 - .5412}{.0024}(10') = 57^\circ 14.2'$$

(*c*) Refer to the table on page 208.

$$\begin{aligned} \tan 49^\circ 10' &= 1.1571 \\ \tan 49^\circ 0' &= \underline{1.1504} \\ \text{Tabular difference} &= .0067 \end{aligned}$$

Then $$\tan^{-1}(1.1536) = 49^\circ 0' + \frac{1.1536 - 1.1504}{.0067}(10') = 49^\circ 4.8'$$

Other inverse trigonometric functions can be obtained similarly.

## TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS (RADIANS)

19. Find (*a*) $\sin(.627)$, (*b*) $\cos(1.056)$, (*c*) $\tan(.153)$.

(*a*) Refer to the table on page 213.

$$\begin{aligned} \sin(.630) &= .58914 \\ \sin(.620) &= \underline{.58104} \\ \text{Tabular difference} &= .00810 \end{aligned}$$

Then $$\sin(.627) = .58104 + \tfrac{7}{10}(.00810) = .58671$$

(*b*) Refer to the table on page 214.

$$\begin{aligned} \cos(1.050) &= .49757 \\ \cos(1.060) &= \underline{.48887} \\ \text{Tabular difference} &= .00870 \end{aligned}$$

Then $$\cos(1.056) = .49757 - \tfrac{6}{10}(.00870) = .49235$$

or $$\cos(1.056) = .48887 + \tfrac{4}{10}(.00870) = .49235$$

(*c*) Refer to the table on page 212.

$$\begin{aligned} \tan(.160) &= .16138 \\ \tan(.150) &= \underline{.15114} \\ \text{Tabular difference} &= .01024 \end{aligned}$$

Then $$\tan(.153) = .15114 + \tfrac{3}{10}(.01024) = .15421$$

Similarly other trigonometric functions are obtained.

**20.** Find $\sin^{-1}(.512)$ in radians.

Refer to the table on page 213.

$$\begin{aligned} \sin(.540) &= .51414 \\ \sin(.530) &= \underline{.50553} \\ \text{Tabular difference} &= .00861 \end{aligned}$$

Then $$\sin^{-1}(.512) = .530 + \frac{.512 - .50553}{.00861}(.01) = .5375 \text{ radians}$$

Similarly the other inverse trigonometric functions are obtained.

## COMMON LOGARITHMS OF TRIGONOMETRIC FUNCTIONS

**21.** Find (*a*) $\log \sin 63°17'$, (*b*) $\log \cos 48°44'$.

(*a*) Refer to the table on page 217.

$$\begin{aligned} \log \sin 63°20' &= 9.9512 - 10 \\ \log \sin 63°10' &= \underline{9.9505 - 10} \\ \text{Tabular difference} &= .0007 \end{aligned}$$

Then $$\log \sin 63°17' = 9.9505 - 10 + \tfrac{7}{10}(.0007) = 9.9510 - 10$$

(*b*) Refer to the table on page 219.

$$\begin{aligned} \log \cos 48°40' &= 9.8198 - 10 \\ \log \cos 48°50' &= \underline{9.8184 - 10} \\ \text{Tabular difference} &= .0014 \end{aligned}$$

Then $$\log \cos 48°44' = 9.8198 - 10 - \tfrac{4}{10}(.0014) = 9.8192 - 10$$

or $$\log \cos 48°44' = 9.8184 - 10 + \tfrac{6}{10}(.0014) = 9.8192 - 10$$

Similarly we can find logarithms of other trigonometric functions. Note that $\log \sec x = -\log \cos x$, $\log \cot x = -\log \tan x$, $\log \csc x = -\log \sin x$.

**22.** If $\log \tan x = 9.6845 - 10$, find $x$.

Refer to the table on page 220.

$$\begin{aligned} \log \tan 25°50' &= 9.6850 - 10 \\ \log \tan 25°40' &= \underline{9.6817 - 10} \\ \text{Tabular difference} &= .0033 \end{aligned}$$

Then $$x = 25°40' + \frac{9.6845 - 9.6817}{.0033}(10') = 25°48.5'$$

## CONVERSION OF DEGREES, MINUTES AND SECONDS TO RADIANS

**23.** Find 75° 28′ 47″ in radians.

Refer to the table on page 223.

$$\begin{aligned} 70° &= 1.221730 \text{ radians} \\ 5° &= .087267 \\ 20' &= .005818 \\ 8' &= .002327 \\ 40'' &= .000194 \\ \underline{7''} &= \underline{.000034} \\ \text{Adding,}\quad 75°\,28'\,47'' &= 1.317370 \text{ radians} \end{aligned}$$

## CONVERSION OF RADIANS TO DEGREES, MINUTES AND SECONDS

**24.** Find 2.547 radians in degrees, minutes and seconds.

Refer to the table on page 222.

$$\begin{array}{lrcl}
 & 2 \text{ radians} & = & 114^\circ\ 35'\ 29.6'' \\
 & .5 \phantom{\text{ radians}} & = & 28^\circ\ 38'\ 52.4'' \\
 & .04 \phantom{\text{ radians}} & = & 2^\circ\ 17'\ 30.6'' \\
 & \underline{.007} \phantom{\text{ radians}} & = & \underline{0^\circ\ 24'\ \ 3.9''} \\
\text{Adding,} & 2.547 \text{ radians} & = & 144^\circ\ 114'\ 116.5'' = 145^\circ\ 55'\ 56.5''
\end{array}$$

## CONVERSION OF RADIANS TO FRACTIONS OF A DEGREE

**25.** Find 1.382 radians in terms of degrees.

Refer to the table on page 222.

$$\begin{array}{lrcl}
 & 1 \text{ radian} & = & 57.2958^\circ \\
 & .3 \phantom{\text{ radian}} & = & 17.1887^\circ \\
 & .08 \phantom{\text{ radian}} & = & 4.5837^\circ \\
 & \underline{.002} \phantom{\text{ radian}} & = & \underline{.1146^\circ} \\
\text{Adding,} & 1.382 \text{ radians} & = & 79.1828^\circ
\end{array}$$

## EXPONENTIAL AND HYPERBOLIC FUNCTIONS

**26.** Find (*a*) $e^{5.24}$, (*b*) $e^{-.158}$.

(*a*) Refer to the table on page 226.

$$\begin{array}{rcl}
e^{5.30} & = & 200.34 \\
e^{5.20} & = & \underline{181.27} \\
\text{Tabular difference} & = & 19.07
\end{array}$$

Then $e^{5.24} = 181.27 + \frac{4}{10}(19.07) = 188.90$

(*b*) Refer to the table on page 227.

$$\begin{array}{rcl}
e^{-.150} & = & .86071 \\
e^{-.160} & = & \underline{.85214} \\
\text{Tabular difference} & = & .00857
\end{array}$$

Then $e^{-.158} = .86071 - \frac{8}{10}(.00857) = .85385$

or $e^{-.158} = .85214 + \frac{2}{10}(.00857) = .85385$

**27.** Find (*a*) $\sinh(4.846)$, (*b*) $\operatorname{sech}(.163)$.

(*a*) Refer to the table on page 229.

$$\begin{array}{rcl}
\sinh(4.850) & = & 63.866 \\
\sinh(4.840) & = & \underline{63.231} \\
\text{Tabular difference} & = & .635
\end{array}$$

Then $\sinh(4.846) = 63.231 + \frac{6}{10}(.635) = 63.612$

(*b*) Refer to the table on page 230.

$$\begin{array}{rcl}
\cosh(.170) & = & 1.0145 \\
\cosh(.160) & = & \underline{1.0128} \\
\text{Tabular difference} & = & .0017
\end{array}$$

Then $\cosh(.163) = 1.0128 + \frac{3}{10}(.0017) = 1.0133$

and so $\operatorname{sech}(.163) = \dfrac{1}{\cosh(.163)} = \dfrac{1}{1.0133} = .98687$

**28.** Find $\tanh^{-1}(.71423)$.

Refer to the table on page 232.

$$\begin{aligned} \tanh(.900) &= .71630 \\ \tanh(.890) &= \underline{.71139} \\ \text{Tabular difference} &= .00491 \end{aligned}$$

Then $$\tanh^{-1}(.71423) = .890 + \frac{.71423 - .71139}{.00491}(10) = .8958$$

## INTEREST AND ANNUITIES

**29.** A man deposits \$2800 in a bank which pays 5% compounded quarterly. What will the deposit amount to in 8 years?

There are $n = 8 \cdot 4 = 32$ payment periods at interest rate $r = .05/4 = .0125$ per period. Then the amount is

$$A = \$2800(1 + .0125)^{32} = \$2800(1.4881) = \$4166.68$$

using the table on page 240.

**30.** A man expects to receive \$12,000 in 10 years. How much is that money worth now, considering interest at 6% compounded semi-annually?

We are asked for the present value $P$ which will amount to $A = \$12{,}000$ in 10 years. Since there are $n = 10 \cdot 2 = 20$ payment periods at interest rate $r = .06/2 = .03$ per period, the present value is

$$P = \$12{,}000(1 + .03)^{-20} = \$12{,}000(.55368) = \$6644.16$$

using the table on page 241.

**31.** An investor has an annuity in which a payment of \$500 is made at the end of each year. If interest is 4% compounded annually, what is the amount of the annuity after 20 years?

Here $r = .04$, $n = 20$ and the amount is [see table on page 242],

$$\$500\left[\frac{(1 + .04)^{20} - 1}{.04}\right] = \$500(29.7781) = \$14{,}889.05$$

**32.** What is the present value of an annuity of \$120 at the end of each 3 months for 12 years at 6% compounded quarterly?

Here $n = 4 \cdot 12 = 48$ payment periods, $r = .06/4 = .015$ and the present value is

$$\$120\left[\frac{1 - (1.015)^{-48}}{.015}\right] = \$120(34.0426) = \$4085.11$$

using the table on page 243.

## TABLE 1

# FOUR PLACE COMMON LOGARITHMS

## $\log_{10} N$ or $\log N$

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Proportional Parts | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 10 | 0000 | 0043 | 0086 | 0128 | 0170 | 0212 | 0253 | 0294 | 0334 | 0374 | 4 | 8 | 12 | 17 | 21 | 25 | 29 | 33 | 37 |
| 11 | 0414 | 0453 | 0492 | 0531 | 0569 | 0607 | 0645 | 0682 | 0719 | 0755 | 4 | 8 | 11 | 15 | 19 | 23 | 26 | 30 | 34 |
| 12 | 0792 | 0828 | 0864 | 0899 | 0934 | 0969 | 1004 | 1038 | 1072 | 1106 | 3 | 7 | 10 | 14 | 17 | 21 | 24 | 28 | 31 |
| 13 | 1139 | 1173 | 1206 | 1239 | 1271 | 1303 | 1335 | 1367 | 1399 | 1430 | 3 | 6 | 10 | 13 | 16 | 19 | 23 | 26 | 29 |
| 14 | 1461 | 1492 | 1523 | 1553 | 1584 | 1614 | 1644 | 1673 | 1703 | 1732 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 |
| 15 | 1761 | 1790 | 1818 | 1847 | 1875 | 1903 | 1931 | 1959 | 1987 | 2014 | 3 | 6 | 8 | 11 | 14 | 17 | 20 | 22 | 25 |
| 16 | 2041 | 2068 | 2095 | 2122 | 2148 | 2175 | 2201 | 2227 | 2253 | 2279 | 3 | 5 | 8 | 11 | 13 | 16 | 18 | 21 | 24 |
| 17 | 2304 | 2330 | 2355 | 2380 | 2405 | 2430 | 2455 | 2480 | 2504 | 2529 | 2 | 5 | 7 | 10 | 12 | 15 | 17 | 20 | 22 |
| 18 | 2553 | 2577 | 2601 | 2625 | 2648 | 2672 | 2695 | 2718 | 2742 | 2765 | 2 | 5 | 7 | 9 | 12 | 14 | 16 | 19 | 21 |
| 19 | 2788 | 2810 | 2833 | 2856 | 2878 | 2900 | 2923 | 2945 | 2967 | 2989 | 2 | 4 | 7 | 9 | 11 | 13 | 16 | 18 | 20 |
| 20 | 3010 | 3032 | 3054 | 3075 | 3096 | 3118 | 3139 | 3160 | 3181 | 3201 | 2 | 4 | 6 | 8 | 11 | 13 | 15 | 17 | 19 |
| 21 | 3222 | 3243 | 3263 | 3284 | 3304 | 3324 | 3345 | 3365 | 3385 | 3404 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| 22 | 3424 | 3444 | 3464 | 3483 | 3502 | 3522 | 3541 | 3560 | 3579 | 3598 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 15 | 17 |
| 23 | 3617 | 3636 | 3655 | 3674 | 3692 | 3711 | 3729 | 3747 | 3766 | 3784 | 2 | 4 | 6 | 7 | 9 | 11 | 13 | 15 | 17 |
| 24 | 3802 | 3820 | 3838 | 3856 | 3874 | 3892 | 3909 | 3927 | 3945 | 3962 | 2 | 4 | 5 | 7 | 9 | 11 | 12 | 14 | 16 |
| 25 | 3979 | 3997 | 4014 | 4031 | 4048 | 4065 | 4082 | 4099 | 4116 | 4133 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 15 |
| 26 | 4150 | 4166 | 4183 | 4200 | 4216 | 4232 | 4249 | 4265 | 4281 | 4298 | 2 | 3 | 5 | 7 | 8 | 10 | 11 | 13 | 15 |
| 27 | 4314 | 4330 | 4346 | 4362 | 4378 | 4393 | 4409 | 4425 | 4440 | 4456 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 13 | 14 |
| 28 | 4472 | 4487 | 4502 | 4518 | 4533 | 4548 | 4564 | 4579 | 4594 | 4609 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| 29 | 4624 | 4639 | 4654 | 4669 | 4683 | 4698 | 4713 | 4728 | 4742 | 4757 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 |
| 30 | 4771 | 4786 | 4800 | 4814 | 4829 | 4843 | 4857 | 4871 | 4886 | 4900 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |
| 31 | 4914 | 4928 | 4942 | 4955 | 4969 | 4983 | 4997 | 5011 | 5024 | 5038 | 1 | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 12 |
| 32 | 5051 | 5065 | 5079 | 5092 | 5105 | 5119 | 5132 | 5145 | 5159 | 5172 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 12 |
| 33 | 5185 | 5198 | 5211 | 5224 | 5237 | 5250 | 5263 | 5276 | 5289 | 5302 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 12 |
| 34 | 5315 | 5328 | 5340 | 5353 | 5366 | 5378 | 5391 | 5403 | 5416 | 5428 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 11 |
| 35 | 5441 | 5453 | 5465 | 5478 | 5490 | 5502 | 5514 | 5527 | 5539 | 5551 | 1 | 2 | 4 | 5 | 6 | 7 | 9 | 10 | 11 |
| 36 | 5563 | 5575 | 5587 | 5599 | 5611 | 5623 | 5635 | 5647 | 5658 | 5670 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| 37 | 5682 | 5694 | 5705 | 5717 | 5729 | 5740 | 5752 | 5763 | 5775 | 5786 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| 38 | 5798 | 5809 | 5821 | 5832 | 5843 | 5855 | 5866 | 5877 | 5888 | 5899 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| 39 | 5911 | 5922 | 5933 | 5944 | 5955 | 5966 | 5977 | 5988 | 5999 | 6010 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 10 |
| 40 | 6021 | 6031 | 6042 | 6053 | 6064 | 6075 | 6085 | 6096 | 6107 | 6117 | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 9 | 10 |
| 41 | 6128 | 6138 | 6149 | 6160 | 6170 | 6180 | 6191 | 6201 | 6212 | 6222 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 42 | 6232 | 6243 | 6253 | 6263 | 6274 | 6284 | 6294 | 6304 | 6314 | 6325 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 43 | 6335 | 6345 | 6355 | 6365 | 6375 | 6385 | 6395 | 6405 | 6415 | 6425 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 44 | 6435 | 6444 | 6454 | 6464 | 6474 | 6484 | 6493 | 6503 | 6513 | 6522 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 45 | 6532 | 6542 | 6551 | 6561 | 6571 | 6580 | 6590 | 6599 | 6609 | 6618 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 46 | 6628 | 6637 | 6646 | 6656 | 6665 | 6675 | 6684 | 6693 | 6702 | 6712 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 7 | 8 |
| 47 | 6721 | 6730 | 6739 | 6749 | 6758 | 6767 | 6776 | 6785 | 6794 | 6803 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 7 | 8 |
| 48 | 6812 | 6821 | 6830 | 6839 | 6848 | 6857 | 6866 | 6875 | 6884 | 6893 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| 49 | 6902 | 6911 | 6920 | 6928 | 6937 | 6946 | 6955 | 6964 | 6972 | 6981 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| 50 | 6990 | 6998 | 7007 | 7016 | 7024 | 7033 | 7042 | 7050 | 7059 | 7067 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| 51 | 7076 | 7084 | 7093 | 7101 | 7110 | 7118 | 7126 | 7135 | 7143 | 7152 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| 52 | 7160 | 7168 | 7177 | 7185 | 7193 | 7202 | 7210 | 7218 | 7226 | 7235 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| 53 | 7243 | 7251 | 7259 | 7267 | 7275 | 7284 | 7292 | 7300 | 7308 | 7316 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| 54 | 7324 | 7332 | 7340 | 7348 | 7356 | 7364 | 7372 | 7380 | 7388 | 7396 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

**Table 1**
**(continued)**

## FOUR PLACE COMMON LOGARITHMS

## $\log_{10} N$ or $\log N$

| $N$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Proportional Parts 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 55 | 7404 | 7412 | 7419 | 7427 | 7435 | 7443 | 7451 | 7459 | 7466 | 7474 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 56 | 7482 | 7490 | 7497 | 7505 | 7513 | 7520 | 7528 | 7536 | 7543 | 7551 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 57 | 7559 | 7566 | 7574 | 7582 | 7589 | 7597 | 7604 | 7612 | 7619 | 7627 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 58 | 7634 | 7642 | 7649 | 7657 | 7664 | 7672 | 7679 | 7686 | 7694 | 7701 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| 59 | 7709 | 7716 | 7723 | 7731 | 7738 | 7745 | 7752 | 7760 | 7767 | 7774 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| **60** | 7782 | 7789 | 7796 | 7803 | 7810 | 7818 | 7825 | 7832 | 7839 | 7846 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| 61 | 7853 | 7860 | 7868 | 7875 | 7882 | 7889 | 7896 | 7903 | 7910 | 7917 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| 62 | 7924 | 7931 | 7938 | 7945 | 7952 | 7959 | 7966 | 7973 | 7980 | 7987 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 6 |
| 63 | 7993 | 8000 | 8007 | 8014 | 8021 | 8028 | 8035 | 8041 | 8048 | 8055 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 64 | 8062 | 8069 | 8075 | 8082 | 8089 | 8096 | 8102 | 8109 | 8116 | 8122 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 65 | 8129 | 8136 | 8142 | 8149 | 8156 | 8162 | 8169 | 8176 | 8182 | 8189 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 66 | 8195 | 8202 | 8209 | 8215 | 8222 | 8228 | 8235 | 8241 | 8248 | 8254 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 67 | 8261 | 8267 | 8274 | 8280 | 8287 | 8293 | 8299 | 8306 | 8312 | 8319 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 68 | 8325 | 8331 | 8338 | 8344 | 8351 | 8357 | 8363 | 8370 | 8376 | 8382 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| 69 | 8388 | 8395 | 8401 | 8407 | 8414 | 8420 | 8426 | 8432 | 8439 | 8445 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| **70** | 8451 | 8457 | 8463 | 8470 | 8476 | 8482 | 8488 | 8494 | 8500 | 8506 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| 71 | 8513 | 8519 | 8525 | 8531 | 8537 | 8543 | 8549 | 8555 | 8561 | 8567 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 72 | 8573 | 8579 | 8585 | 8591 | 8597 | 8603 | 8609 | 8615 | 8621 | 8627 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 73 | 8633 | 8639 | 8645 | 8651 | 8657 | 8663 | 8669 | 8675 | 8681 | 8686 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 74 | 8692 | 8698 | 8704 | 8710 | 8716 | 8722 | 8727 | 8733 | 8739 | 8745 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 75 | 8751 | 8756 | 8762 | 8768 | 8774 | 8779 | 8785 | 8791 | 8797 | 8802 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| 76 | 8808 | 8814 | 8820 | 8825 | 8831 | 8837 | 8842 | 8848 | 8854 | 8859 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| 77 | 8865 | 8871 | 8876 | 8882 | 8887 | 8893 | 8899 | 8904 | 8910 | 8915 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 78 | 8921 | 8927 | 8932 | 8938 | 8943 | 8949 | 8954 | 8960 | 8965 | 8971 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 79 | 8976 | 8982 | 8987 | 8993 | 8998 | 9004 | 9009 | 9015 | 9020 | 9025 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| **80** | 9031 | 9036 | 9042 | 9047 | 9053 | 9058 | 9063 | 9069 | 9074 | 9079 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 81 | 9085 | 9090 | 9096 | 9101 | 9106 | 9112 | 9117 | 9122 | 9128 | 9133 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 82 | 9138 | 9143 | 9149 | 9154 | 9159 | 9165 | 9170 | 9175 | 9180 | 9186 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 83 | 9191 | 9196 | 9201 | 9206 | 9212 | 9217 | 9222 | 9227 | 9232 | 9238 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 84 | 9243 | 9248 | 9253 | 9258 | 9263 | 9269 | 9274 | 9279 | 9284 | 9289 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 85 | 9294 | 9299 | 9304 | 9309 | 9315 | 9320 | 9325 | 9330 | 9335 | 9340 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 86 | 9345 | 9350 | 9355 | 9360 | 9365 | 9370 | 9375 | 9380 | 9385 | 9390 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 87 | 9395 | 9400 | 9405 | 9410 | 9415 | 9420 | 9425 | 9430 | 9435 | 9440 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 88 | 9445 | 9450 | 9455 | 9460 | 9465 | 9469 | 9474 | 9479 | 9484 | 9489 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 89 | 9494 | 9499 | 9504 | 9509 | 9513 | 9518 | 9523 | 9528 | 9533 | 9538 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **90** | 9542 | 9547 | 9552 | 9557 | 9562 | 9566 | 9571 | 9576 | 9581 | 9586 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 91 | 9590 | 9595 | 9600 | 9605 | 9609 | 9614 | 9619 | 9624 | 9628 | 9633 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 92 | 9638 | 9643 | 9647 | 9652 | 9657 | 9661 | 9666 | 9671 | 9675 | 9680 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 93 | 9685 | 9689 | 9694 | 9699 | 9703 | 9708 | 9713 | 9717 | 9722 | 9727 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 94 | 9731 | 9736 | 9741 | 9745 | 9750 | 9754 | 9759 | 9763 | 9768 | 9773 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 95 | 9777 | 9782 | 9786 | 9791 | 9795 | 9800 | 9805 | 9809 | 9814 | 9818 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 96 | 9823 | 9827 | 9832 | 9836 | 9841 | 9845 | 9850 | 9854 | 9859 | 9863 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 97 | 9868 | 9872 | 9877 | 9881 | 9886 | 9890 | 9894 | 9899 | 9903 | 9908 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 98 | 9912 | 9917 | 9921 | 9926 | 9930 | 9934 | 9939 | 9943 | 9948 | 9952 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 99 | 9956 | 9961 | 9965 | 9969 | 9974 | 9978 | 9983 | 9987 | 9991 | 9996 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| $N$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

**TABLE**

**2**

## FOUR PLACE COMMON ANTILOGARITHMS

## $10^p$ or antilog $p$

| $p$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Proportional Parts | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **.00** | 1000 | 1002 | 1005 | 1007 | 1009 | 1012 | 1014 | 1016 | 1019 | 1021 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| .01 | 1023 | 1026 | 1028 | 1030 | 1033 | 1035 | 1038 | 1040 | 1042 | 1045 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| .02 | 1047 | 1050 | 1052 | 1054 | 1057 | 1059 | 1062 | 1064 | 1067 | 1069 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| .03 | 1072 | 1074 | 1076 | 1079 | 1081 | 1084 | 1086 | 1089 | 1091 | 1094 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| .04 | 1096 | 1099 | 1102 | 1104 | 1107 | 1109 | 1112 | 1114 | 1117 | 1119 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| .05 | 1122 | 1125 | 1127 | 1130 | 1132 | 1135 | 1138 | 1140 | 1143 | 1146 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| .06 | 1148 | 1151 | 1153 | 1156 | 1159 | 1161 | 1164 | 1167 | 1169 | 1172 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| .07 | 1175 | 1178 | 1180 | 1183 | 1186 | 1189 | 1191 | 1194 | 1197 | 1199 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| .08 | 1202 | 1205 | 1208 | 1211 | 1213 | 1216 | 1219 | 1222 | 1225 | 1227 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| .09 | 1230 | 1233 | 1236 | 1239 | 1242 | 1245 | 1247 | 1250 | 1253 | 1256 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| **.10** | 1259 | 1262 | 1265 | 1268 | 1271 | 1274 | 1276 | 1279 | 1282 | 1285 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| .11 | 1288 | 1291 | 1294 | 1297 | 1300 | 1303 | 1306 | 1309 | 1312 | 1315 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 |
| .12 | 1318 | 1321 | 1324 | 1327 | 1330 | 1334 | 1337 | 1340 | 1343 | 1346 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 |
| .13 | 1349 | 1352 | 1355 | 1358 | 1361 | 1365 | 1368 | 1371 | 1374 | 1377 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| .14 | 1380 | 1384 | 1387 | 1390 | 1393 | 1396 | 1400 | 1403 | 1406 | 1409 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| .15 | 1413 | 1416 | 1419 | 1422 | 1426 | 1429 | 1432 | 1435 | 1439 | 1442 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| .16 | 1445 | 1449 | 1452 | 1455 | 1459 | 1462 | 1466 | 1469 | 1472 | 1476 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| .17 | 1479 | 1483 | 1486 | 1489 | 1493 | 1496 | 1500 | 1503 | 1507 | 1510 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| .18 | 1514 | 1517 | 1521 | 1524 | 1528 | 1531 | 1535 | 1538 | 1542 | 1545 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| .19 | 1549 | 1552 | 1556 | 1560 | 1563 | 1567 | 1570 | 1574 | 1578 | 1581 | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 |
| **.20** | 1585 | 1589 | 1592 | 1596 | 1600 | 1603 | 1607 | 1611 | 1614 | 1618 | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 |
| .21 | 1622 | 1626 | 1629 | 1633 | 1637 | 1641 | 1644 | 1648 | 1652 | 1656 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 |
| .22 | 1660 | 1663 | 1667 | 1671 | 1675 | 1679 | 1683 | 1687 | 1690 | 1694 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 |
| .23 | 1698 | 1702 | 1706 | 1710 | 1714 | 1718 | 1722 | 1726 | 1730 | 1734 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| .24 | 1738 | 1742 | 1746 | 1750 | 1754 | 1758 | 1762 | 1766 | 1770 | 1774 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| .25 | 1778 | 1782 | 1786 | 1791 | 1795 | 1799 | 1803 | 1807 | 1811 | 1816 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| .26 | 1820 | 1824 | 1828 | 1832 | 1837 | 1841 | 1845 | 1849 | 1854 | 1858 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| .27 | 1862 | 1866 | 1871 | 1875 | 1879 | 1884 | 1888 | 1892 | 1897 | 1901 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| .28 | 1905 | 1910 | 1914 | 1919 | 1923 | 1928 | 1932 | 1936 | 1941 | 1945 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| .29 | 1950 | 1954 | 1959 | 1963 | 1968 | 1972 | 1977 | 1982 | 1986 | 1991 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **.30** | 1995 | 2000 | 2004 | 2009 | 2014 | 2018 | 2023 | 2028 | 2032 | 2037 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| .31 | 2042 | 2046 | 2051 | 2056 | 2061 | 2065 | 2070 | 2075 | 2080 | 2084 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| .32 | 2089 | 2094 | 2099 | 2104 | 2109 | 2113 | 2118 | 2123 | 2128 | 2133 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| .33 | 2138 | 2143 | 2148 | 2153 | 2158 | 2163 | 2168 | 2173 | 2178 | 2183 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| .34 | 2188 | 2193 | 2198 | 2203 | 2208 | 2213 | 2218 | 2223 | 2228 | 2234 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| .35 | 2239 | 2244 | 2249 | 2254 | 2259 | 2265 | 2270 | 2275 | 2280 | 2286 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| .36 | 2291 | 2296 | 2301 | 2307 | 2312 | 2317 | 2323 | 2328 | 2333 | 2339 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| .37 | 2344 | 2350 | 2355 | 2360 | 2366 | 2371 | 2377 | 2382 | 2388 | 2393 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| .38 | 2399 | 2404 | 2410 | 2415 | 2421 | 2427 | 2432 | 2438 | 2443 | 2449 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| .39 | 2455 | 2460 | 2466 | 2472 | 2477 | 2483 | 2489 | 2495 | 2500 | 2506 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| **.40** | 2512 | 2518 | 2523 | 2529 | 2535 | 2541 | 2547 | 2553 | 2559 | 2564 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| .41 | 2570 | 2576 | 2582 | 2588 | 2594 | 2600 | 2606 | 2612 | 2618 | 2624 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| .42 | 2630 | 2636 | 2642 | 2649 | 2655 | 2661 | 2667 | 2673 | 2679 | 2685 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| .43 | 2692 | 2698 | 2704 | 2710 | 2716 | 2723 | 2729 | 2735 | 2742 | 2748 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| .44 | 2754 | 2761 | 2767 | 2773 | 2780 | 2786 | 2793 | 2799 | 2805 | 2812 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| .45 | 2818 | 2825 | 2831 | 2838 | 2844 | 2851 | 2858 | 2864 | 2871 | 2877 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| .46 | 2884 | 2891 | 2897 | 2904 | 2911 | 2917 | 2924 | 2931 | 2938 | 2944 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| .47 | 2951 | 2958 | 2965 | 2972 | 2979 | 2985 | 2992 | 2999 | 3006 | 3013 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| .48 | 3020 | 3027 | 3034 | 3041 | 3048 | 3055 | 3062 | 3069 | 3076 | 3083 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| .49 | 3090 | 3097 | 3105 | 3112 | 3119 | 3126 | 3133 | 3141 | 3148 | 3155 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| $p$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

**Table 2**
(continued)

## FOUR PLACE COMMON ANTILOGARITHMS

## $10^p$ or antilog $p$

| $p$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Proportional Parts | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| .50 | 3162 | 3170 | 3177 | 3184 | 3192 | 3199 | 3206 | 3214 | 3221 | 3228 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| .51 | 3236 | 3243 | 3251 | 3258 | 3266 | 3273 | 3281 | 3289 | 3296 | 3304 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| .52 | 3311 | 3319 | 3327 | 3334 | 3342 | 3350 | 3357 | 3365 | 3373 | 3381 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| .53 | 3388 | 3396 | 3404 | 3412 | 3420 | 3428 | 3436 | 3443 | 3451 | 3459 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| .54 | 3467 | 3475 | 3483 | 3491 | 3499 | 3508 | 3516 | 3524 | 3532 | 3540 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| .55 | 3548 | 3556 | 3565 | 3573 | 3581 | 3589 | 3597 | 3606 | 3614 | 3622 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| .56 | 3631 | 3639 | 3648 | 3656 | 3664 | 3673 | 3681 | 3690 | 3698 | 3707 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| .57 | 3715 | 3724 | 3733 | 3741 | 3750 | 3758 | 3767 | 3776 | 3784 | 3793 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| .58 | 3802 | 3811 | 3819 | 3828 | 3837 | 3846 | 3855 | 3864 | 3873 | 3882 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| .59 | 3890 | 3899 | 3908 | 3917 | 3926 | 3936 | 3945 | 3954 | 3963 | 3972 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 7 | 8 |
| .60 | 3981 | 3990 | 3999 | 4009 | 4018 | 4027 | 4036 | 4046 | 4055 | 4064 | 1 | 2 | 3 | 4 | 5 | 6 | 6 | 7 | 8 |
| .61 | 4074 | 4083 | 4093 | 4102 | 4111 | 4121 | 4130 | 4140 | 4150 | 4159 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| .62 | 4169 | 4178 | 4188 | 4198 | 4207 | 4217 | 4227 | 4236 | 4246 | 4256 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| .63 | 4266 | 4276 | 4285 | 4295 | 4305 | 4315 | 4325 | 4335 | 4345 | 4355 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| .64 | 4365 | 4375 | 4385 | 4395 | 4406 | 4416 | 4426 | 4436 | 4446 | 4457 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| .65 | 4467 | 4477 | 4487 | 4498 | 4508 | 4519 | 4529 | 4539 | 4550 | 4560 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| .66 | 4571 | 4581 | 4592 | 4603 | 4613 | 4624 | 4634 | 4645 | 4656 | 4667 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 9 | 10 |
| .67 | 4677 | 4688 | 4699 | 4710 | 4721 | 4732 | 4742 | 4753 | 4764 | 4775 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 10 |
| .68 | 4786 | 4797 | 4808 | 4819 | 4831 | 4842 | 4853 | 4864 | 4875 | 4887 | 1 | 2 | 3 | 4 | 6 | 7 | 8 | 9 | 10 |
| .69 | 4898 | 4909 | 4920 | 4932 | 4943 | 4955 | 4966 | 4977 | 4989 | 5000 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| .70 | 5012 | 5023 | 5035 | 5047 | 5058 | 5070 | 5082 | 5093 | 5105 | 5117 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 9 | 11 |
| .71 | 5129 | 5140 | 5152 | 5164 | 5176 | 5188 | 5200 | 5212 | 5224 | 5236 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| .72 | 5248 | 5260 | 5272 | 5284 | 5297 | 5309 | 5321 | 5333 | 5346 | 5358 | 1 | 2 | 4 | 5 | 6 | 7 | 9 | 10 | 11 |
| .73 | 5370 | 5383 | 5395 | 5408 | 5420 | 5433 | 5445 | 5458 | 5470 | 5483 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 11 |
| .74 | 5495 | 5508 | 5521 | 5534 | 5546 | 5559 | 5572 | 5585 | 5598 | 5610 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 12 |
| .75 | 5623 | 5636 | 5649 | 5662 | 5675 | 5689 | 5702 | 5715 | 5728 | 5741 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 10 | 12 |
| .76 | 5754 | 5768 | 5781 | 5794 | 5808 | 5821 | 5834 | 5848 | 5861 | 5875 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 12 |
| .77 | 5888 | 5902 | 5916 | 5929 | 5943 | 5957 | 5970 | 5984 | 5998 | 6012 | 1 | 3 | 4 | 5 | 7 | 8 | 10 | 11 | 12 |
| .78 | 6026 | 6039 | 6053 | 6067 | 6081 | 6095 | 6109 | 6124 | 6138 | 6152 | 1 | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 13 |
| .79 | 6166 | 6180 | 6194 | 6209 | 6223 | 6237 | 6252 | 6266 | 6281 | 6295 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |
| .80 | 6310 | 6324 | 6339 | 6353 | 6368 | 6383 | 6397 | 6412 | 6427 | 6442 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 |
| .81 | 6457 | 6471 | 6486 | 6501 | 6516 | 6531 | 6546 | 6561 | 6577 | 6592 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| .82 | 6607 | 6622 | 6637 | 6653 | 6668 | 6683 | 6699 | 6714 | 6730 | 6745 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| .83 | 6761 | 6776 | 6792 | 6808 | 6823 | 6839 | 6855 | 6871 | 6887 | 6902 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 13 | 14 |
| .84 | 6918 | 6934 | 6950 | 6966 | 6982 | 6998 | 7015 | 7031 | 7047 | 7063 | 2 | 3 | 5 | 6 | 8 | 10 | 11 | 13 | 15 |
| .85 | 7079 | 7096 | 7112 | 7129 | 7145 | 7161 | 7178 | 7194 | 7211 | 7228 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 13 | 15 |
| .86 | 7244 | 7261 | 7278 | 7295 | 7311 | 7328 | 7345 | 7362 | 7379 | 7396 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 13 | 15 |
| .87 | 7413 | 7430 | 7447 | 7464 | 7482 | 7499 | 7516 | 7534 | 7551 | 7568 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 16 |
| .88 | 7586 | 7603 | 7621 | 7638 | 7656 | 7674 | 7691 | 7709 | 7727 | 7745 | 2 | 4 | 5 | 7 | 9 | 11 | 12 | 14 | 16 |
| .89 | 7762 | 7780 | 7798 | 7816 | 7834 | 7852 | 7870 | 7889 | 7907 | 7925 | 2 | 4 | 5 | 7 | 9 | 11 | 13 | 14 | 16 |
| .90 | 7943 | 7962 | 7980 | 7998 | 8017 | 8035 | 8054 | 8072 | 8091 | 8110 | 2 | 4 | 6 | 7 | 9 | 11 | 13 | 15 | 17 |
| .91 | 8128 | 8147 | 8166 | 8185 | 8204 | 8222 | 8241 | 8260 | 8279 | 8299 | 2 | 4 | 6 | 8 | 9 | 11 | 13 | 15 | 17 |
| .92 | 8318 | 8337 | 8356 | 8375 | 8395 | 8414 | 8433 | 8453 | 8472 | 8492 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 15 | 17 |
| .93 | 8511 | 8531 | 8551 | 8570 | 8590 | 8610 | 8630 | 8650 | 8670 | 8690 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| .94 | 8710 | 8730 | 8750 | 8770 | 8790 | 8810 | 8831 | 8851 | 8872 | 8892 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| .95 | 8913 | 8933 | 8954 | 8974 | 8995 | 9016 | 9036 | 9057 | 9078 | 9099 | 2 | 4 | 6 | 8 | 10 | 12 | 15 | 17 | 19 |
| .96 | 9120 | 9141 | 9162 | 9183 | 9204 | 9226 | 9247 | 9268 | 9290 | 9311 | 2 | 4 | 6 | 8 | 11 | 13 | 15 | 17 | 19 |
| .97 | 9333 | 9354 | 9376 | 9397 | 9419 | 9441 | 9462 | 9484 | 9506 | 9528 | 2 | 4 | 7 | 9 | 11 | 13 | 15 | 17 | 20 |
| .98 | 9550 | 9572 | 9594 | 9616 | 9638 | 9661 | 9683 | 9705 | 9727 | 9750 | 2 | 4 | 7 | 9 | 11 | 13 | 16 | 18 | 20 |
| .99 | 9772 | 9795 | 9817 | 9840 | 9863 | 9886 | 9908 | 9931 | 9954 | 9977 | 2 | 5 | 7 | 9 | 11 | 14 | 16 | 18 | 20 |
| $p$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

**TABLE**

**3**

## Sin *x* ($x$ in degrees and minutes)

| $x$ | 0′ | 10′ | 20′ | 30′ | 40′ | 50′ |
|---|---|---|---|---|---|---|
| 0° | .0000 | .0029 | .0058 | .0087 | .0116 | .0145 |
| 1 | .0175 | .0204 | .0233 | .0262 | .0291 | .0320 |
| 2 | .0349 | .0378 | .0407 | .0436 | .0465 | .0494 |
| 3 | .0523 | .0552 | .0581 | .0610 | .0640 | .0669 |
| 4 | .0698 | .0727 | .0756 | .0785 | .0814 | .0843 |
| 5° | .0872 | .0901 | .0929 | .0958 | .0987 | .1016 |
| 6 | .1045 | .1074 | .1103 | .1132 | .1161 | .1190 |
| 7 | .1219 | .1248 | .1276 | .1305 | .1334 | .1363 |
| 8 | .1392 | .1421 | .1449 | .1478 | .1507 | .1536 |
| 9 | .1564 | .1593 | .1622 | .1650 | .1679 | .1708 |
| 10° | .1736 | .1765 | .1794 | .1822 | .1851 | .1880 |
| 11 | .1908 | .1937 | .1965 | .1994 | .2022 | .2051 |
| 12 | .2079 | .2108 | .2136 | .2164 | .2193 | .2221 |
| 13 | .2250 | .2278 | .2306 | .2334 | .2363 | .2391 |
| 14 | .2419 | .2447 | .2476 | .2504 | .2532 | .2560 |
| 15° | .2588 | .2616 | .2644 | .2672 | .2700 | .2728 |
| 16 | .2756 | .2784 | .2812 | .2840 | .2868 | .2896 |
| 17 | .2924 | .2952 | .2979 | .3007 | .3035 | .3062 |
| 18 | .3090 | .3118 | .3145 | .3173 | .3201 | .3228 |
| 19 | .3256 | .3283 | .3311 | .3338 | .3365 | .3393 |
| 20° | .3420 | .3448 | .3475 | .3502 | .3529 | .3557 |
| 21 | .3584 | .3611 | .3638 | .3665 | .3692 | .3719 |
| 22 | .3746 | .3773 | .3800 | .3827 | .3854 | .3881 |
| 23 | .3907 | .3934 | .3961 | .3987 | .4014 | .4041 |
| 24 | .4067 | .4094 | .4120 | .4147 | .4173 | .4200 |
| 25° | .4226 | .4253 | .4279 | .4305 | .4331 | .4358 |
| 26 | .4384 | .4410 | .4436 | .4462 | .4488 | .4514 |
| 27 | .4540 | .4566 | .4592 | .4617 | .4643 | .4669 |
| 28 | .4695 | .4720 | .4746 | .4772 | .4797 | .4823 |
| 29 | .4848 | .4874 | .4899 | .4924 | .4950 | .4975 |
| 30° | .5000 | .5025 | .5050 | .5075 | .5100 | .5125 |
| 31 | .5150 | .5175 | .5200 | .5225 | .5250 | .5275 |
| 32 | .5299 | .5324 | .5348 | .5373 | .5398 | .5422 |
| 33 | .5446 | .5471 | .5495 | .5519 | .5544 | .5568 |
| 34 | .5592 | .5616 | .5640 | .5664 | .5688 | .5712 |
| 35° | .5736 | .5760 | .5783 | .5807 | .5831 | .5854 |
| 36 | .5878 | .5901 | .5925 | .5948 | .5972 | .5995 |
| 37 | .6018 | .6041 | .6065 | .6088 | .6111 | .6134 |
| 38 | .6157 | .6180 | .6202 | .6225 | .6248 | .6271 |
| 39 | .6293 | .6316 | .6338 | .6361 | .6383 | .6406 |
| 40° | .6428 | .6450 | .6472 | .6494 | .6517 | .6539 |
| 41 | .6561 | .6583 | .6604 | .6626 | .6648 | .6670 |
| 42 | .6691 | .6713 | .6734 | .6756 | .6777 | .6799 |
| 43 | .6820 | .6841 | .6862 | .6884 | .6905 | .6926 |
| 44 | .6947 | .6967 | .6988 | .7009 | .7030 | .7050 |
| 45° | .7071 | .7092 | .7112 | .7133 | .7153 | .7173 |

| $x$ | 0′ | 10′ | 20′ | 30′ | 40′ | 50′ |
|---|---|---|---|---|---|---|
| 45° | .7071 | .7092 | .7112 | .7133 | .7153 | .7173 |
| 46 | .7193 | .7214 | .7234 | .7254 | .7274 | .7294 |
| 47 | .7314 | .7333 | .7353 | .7373 | .7392 | .7412 |
| 48 | .7431 | .7451 | .7470 | .7490 | .7509 | .7528 |
| 49 | .7547 | .7566 | .7585 | .7604 | .7623 | .7642 |
| 50° | .7660 | .7679 | .7698 | .7716 | .7735 | .7753 |
| 51 | .7771 | .7790 | .7808 | .7826 | .7844 | .7862 |
| 52 | .7880 | .7898 | .7916 | .7934 | .7951 | .7969 |
| 53 | .7986 | .8004 | .8021 | .8039 | .8056 | .8073 |
| 54 | .8090 | .8107 | .8124 | .8141 | .8158 | .8175 |
| 55° | .8192 | .8208 | .8225 | .8241 | .8258 | .8274 |
| 56 | .8290 | .8307 | .8323 | .8339 | .8355 | .8371 |
| 57 | .8387 | .8403 | .8418 | .8434 | .8450 | .8465 |
| 58 | .8480 | .8496 | .8511 | .8526 | .8542 | .8557 |
| 59 | .8572 | .8587 | .8601 | .8616 | .8631 | .8646 |
| 60° | .8660 | .8675 | .8689 | .8704 | .8718 | .8732 |
| 61 | .8746 | .8760 | .8774 | .8788 | .8802 | .8816 |
| 62 | .8829 | .8843 | .8857 | .8870 | .8884 | .8897 |
| 63 | .8910 | .8923 | .8936 | .8949 | .8962 | .8975 |
| 64 | .8988 | .9001 | .9013 | .9026 | .9038 | .9051 |
| 65° | .9063 | .9075 | .9088 | .9100 | .9112 | .9124 |
| 66 | .9135 | .9147 | .9159 | .9171 | .9182 | .9194 |
| 67 | .9205 | .9216 | .9228 | .9239 | .9250 | .9261 |
| 68 | .9272 | .9283 | .9293 | .9304 | .9315 | .9325 |
| 69 | .9336 | .9346 | .9356 | .9367 | .9377 | .9387 |
| 70° | .9397 | .9407 | .9417 | .9426 | .9436 | .9446 |
| 71 | .9455 | .9465 | .9474 | .9483 | .9492 | .9502 |
| 72 | .9511 | .9520 | .9528 | .9537 | .9546 | .9555 |
| 73 | .9563 | .9572 | .9580 | .9588 | .9596 | .9605 |
| 74 | .9613 | .9621 | .9628 | .9636 | .9644 | .9652 |
| 75° | .9659 | .9667 | .9674 | .9681 | .9689 | .9696 |
| 76 | .9703 | .9710 | .9717 | .9724 | .9730 | .9737 |
| 77 | .9744 | .9750 | .9757 | .9763 | .9769 | .9775 |
| 78 | .9781 | .9787 | .9793 | .9799 | .9805 | .9811 |
| 79 | .9816 | .9822 | .9827 | .9833 | .9838 | .9843 |
| 80° | .9848 | .9853 | .9858 | .9863 | .9868 | .9872 |
| 81 | .9877 | .9881 | .9886 | .9890 | .9894 | .9899 |
| 82 | .9903 | .9907 | .9911 | .9914 | .9918 | .9922 |
| 83 | .9925 | .9929 | .9932 | .9936 | .9939 | .9942 |
| 84 | .9945 | .9948 | .9951 | .9954 | .9957 | .9959 |
| 85° | .9962 | .9964 | .9967 | .9969 | .9971 | .9974 |
| 86 | .9976 | .9978 | .9980 | .9981 | .9983 | .9985 |
| 87 | .9986 | .9988 | .9989 | .9990 | .9992 | .9993 |
| 88 | .9994 | .9995 | .9996 | .9997 | .9997 | .9998 |
| 89 | .9998 | .9999 | .9999 | 1.0000 | 1.0000 | 1.0000 |
| 90° | 1.0000 |  |  |  |  |  |

## TABLE 4

## Cos $x$ ($x$ in degrees and minutes)

| $x$ | 0′ | 10′ | 20′ | 30′ | 40′ | 50′ |
|---|---|---|---|---|---|---|
| 0° | 1.0000 | 1.0000 | 1.0000 | 1.0000 | .9999 | .9999 |
| 1 | .9998 | .9998 | .9997 | .9997 | .9996 | .9995 |
| 2 | .9994 | .9993 | .9992 | .9990 | .9989 | .9988 |
| 3 | .9986 | .9985 | .9983 | .9981 | .9980 | .9978 |
| 4 | .9976 | .9974 | .9971 | .9969 | .9967 | .9964 |
| 5° | .9962 | .9959 | .9957 | .9954 | .9951 | .9948 |
| 6 | .9945 | .9942 | .9939 | .9936 | .9932 | .9929 |
| 7 | .9925 | .9922 | .9918 | .9914 | .9911 | .9907 |
| 8 | .9903 | .9899 | .9894 | .9890 | .9886 | .9881 |
| 9 | .9877 | .9872 | .9868 | .9863 | .9858 | .9853 |
| 10° | .9848 | .9843 | .9838 | .9833 | .9827 | .9822 |
| 11 | .9816 | .9811 | .9805 | .9799 | .9793 | .9787 |
| 12 | .9781 | .9775 | .9769 | .9763 | .9757 | .9750 |
| 13 | .9744 | .9737 | .9730 | .9724 | .9717 | .9710 |
| 14 | .9703 | .9696 | .9689 | .9681 | .9674 | .9667 |
| 15° | .9659 | .9652 | .9644 | .9636 | .9628 | .9621 |
| 16 | .9613 | .9605 | .9596 | .9588 | .9580 | .9572 |
| 17 | .9563 | .9555 | .9546 | .9537 | .9528 | .9520 |
| 18 | .9511 | .9502 | .9492 | .9483 | .9474 | .9465 |
| 19 | .9455 | .9446 | .9436 | .9426 | .9417 | .9407 |
| 20° | .9397 | .9387 | .9377 | .9367 | .9356 | .9346 |
| 21 | .9336 | .9325 | .9315 | .9304 | .9293 | .9283 |
| 22 | .9272 | .9261 | .9250 | .9239 | .9228 | .9216 |
| 23 | .9205 | .9194 | .9182 | .9171 | .9159 | .9147 |
| 24 | .9135 | .9124 | .9112 | .9100 | .9088 | .9075 |
| 25° | .9063 | .9051 | .9038 | .9026 | .9013 | .9001 |
| 26 | .8988 | .8975 | .8962 | .8949 | .8936 | .8923 |
| 27 | .8910 | .8897 | .8884 | .8870 | .8857 | .8843 |
| 28 | .8829 | .8816 | .8802 | .8788 | .8774 | .8760 |
| 29 | .8746 | .8732 | .8718 | .8704 | .8689 | .8675 |
| 30° | .8660 | .8646 | .8631 | .8616 | .8601 | .8587 |
| 31 | .8572 | .8557 | .8542 | .8526 | .8511 | .8496 |
| 32 | .8480 | .8465 | .8450 | .8434 | .8418 | .8403 |
| 33 | .8387 | .8371 | .8355 | .8339 | .8323 | .8307 |
| 34 | .8290 | .8274 | .8258 | .8241 | .8225 | .8208 |
| 35° | .8192 | .8175 | .8158 | .8141 | .8124 | .8107 |
| 36 | .8090 | .8073 | .8056 | .8039 | .8021 | .8004 |
| 37 | .7986 | .7969 | .7951 | .7934 | .7916 | .7898 |
| 38 | .7880 | .7862 | .7844 | .7826 | .7808 | .7790 |
| 39 | .7771 | .7753 | .7735 | .7716 | .7698 | .7679 |
| 40° | .7660 | .7642 | .7623 | .7604 | .7585 | .7566 |
| 41 | .7547 | .7528 | .7509 | .7490 | .7470 | .7451 |
| 42 | .7431 | .7412 | .7392 | .7373 | .7353 | .7333 |
| 43 | .7314 | .7294 | .7274 | .7254 | .7234 | .7214 |
| 44 | .7193 | .7173 | .7153 | .7133 | .7112 | .7092 |
| 45° | .7071 | .7050 | .7030 | .7009 | .6988 | .6967 |

| $x$ | 0′ | 10′ | 20′ | 30′ | 40′ | 50′ |
|---|---|---|---|---|---|---|
| 45° | .7071 | .7050 | .7030 | .7009 | .6988 | .6967 |
| 46 | .6947 | .6926 | .6905 | .6884 | .6862 | .6841 |
| 47 | .6820 | .6799 | .6777 | .6756 | .6734 | .6713 |
| 48 | .6691 | .6670 | .6648 | .6626 | .6604 | .6583 |
| 49 | .6561 | .6539 | .6517 | .6494 | .6472 | .6450 |
| 50° | .6428 | .6406 | .6383 | .6361 | .6338 | .6316 |
| 51 | .6293 | .6271 | .6248 | .6225 | .6202 | .6180 |
| 52 | .6157 | .6134 | .6111 | .6088 | .6065 | .6041 |
| 53 | .6018 | .5995 | .5972 | .5948 | .5925 | .5901 |
| 54 | .5878 | .5854 | .5831 | .5807 | .5783 | .5760 |
| 55° | .5736 | .5712 | .5688 | .5664 | .5640 | .5616 |
| 56 | .5592 | .5568 | .5544 | .5519 | .5495 | .5471 |
| 57 | .5446 | .5422 | .5398 | .5373 | .5348 | .5324 |
| 58 | .5299 | .5275 | .5250 | .5225 | .5200 | .5175 |
| 59 | .5150 | .5125 | .5100 | .5075 | .5050 | .5025 |
| 60° | .5000 | .4975 | .4950 | .4924 | .4899 | .4874 |
| 61 | .4848 | .4823 | .4797 | .4772 | .4746 | .4720 |
| 62 | .4695 | .4669 | .4643 | .4617 | .4592 | .4566 |
| 63 | .4540 | .4514 | .4488 | .4462 | .4436 | .4410 |
| 64 | .4384 | .4358 | .4331 | .4305 | .4279 | .4253 |
| 65° | .4226 | .4200 | .4173 | .4147 | .4120 | .4094 |
| 66 | .4067 | .4041 | .4014 | .3987 | .3961 | .3934 |
| 67 | .3907 | .3881 | .3854 | .3827 | .3800 | .3773 |
| 68 | .3746 | .3719 | .3692 | .3665 | .3638 | .3611 |
| 69 | .3584 | .3557 | .3529 | .3502 | .3475 | .3448 |
| 70° | .3420 | .3393 | .3365 | .3338 | .3311 | .3283 |
| 71 | .3256 | .3228 | .3201 | .3173 | .3145 | .3118 |
| 72 | .3090 | .3062 | .3035 | .3007 | .2979 | .2952 |
| 73 | .2924 | .2896 | .2868 | .2840 | .2812 | .2784 |
| 74 | .2756 | .2728 | .2700 | .2672 | .2644 | .2616 |
| 75° | .2588 | .2560 | .2532 | .2504 | .2476 | .2447 |
| 76 | .2419 | .2391 | .2363 | .2334 | .2306 | .2278 |
| 77 | .2250 | .2221 | .2193 | .2164 | .2136 | .2108 |
| 78 | .2079 | .2051 | .2022 | .1994 | .1965 | .1937 |
| 79 | .1908 | .1880 | .1851 | .1822 | .1794 | .1765 |
| 80° | .1736 | .1708 | .1679 | .1650 | .1622 | .1593 |
| 81 | .1564 | .1536 | .1507 | .1478 | .1449 | .1421 |
| 82 | .1392 | .1363 | .1334 | .1305 | .1276 | .1248 |
| 83 | .1219 | .1190 | .1161 | .1132 | .1103 | .1074 |
| 84 | .1045 | .1016 | .0987 | .0958 | .0929 | .0901 |
| 85° | .0872 | .0843 | .0814 | .0785 | .0756 | .0727 |
| 86 | .0698 | .0669 | .0640 | .0610 | .0581 | .0552 |
| 87 | .0523 | .0494 | .0465 | .0436 | .0407 | .0378 |
| 88 | .0349 | .0320 | .0291 | .0262 | .0233 | .0204 |
| 89 | .0175 | .0145 | .0116 | .0087 | .0058 | .0029 |
| 90° | .0000 | | | | | |

# TABLE 5

# Tan $x$ ($x$ in degrees and minutes)

| $x$ | 0′ | 10′ | 20′ | 30′ | 40′ | 50′ |
|---|---|---|---|---|---|---|
| 0° | .0000 | .0029 | .0058 | .0087 | .0116 | .0145 |
| 1 | .0175 | .0204 | .0233 | .0262 | .0291 | .0320 |
| 2 | .0349 | .0378 | .0407 | .0437 | .0466 | .0495 |
| 3 | .0524 | .0553 | .0582 | .0612 | .0641 | .0670 |
| 4 | .0699 | .0729 | .0758 | .0787 | .0816 | .0846 |
| 5° | .0875 | .0904 | .0934 | .0963 | .0992 | .1022 |
| 6 | .1051 | .1080 | .1110 | .1139 | .1169 | .1198 |
| 7 | .1228 | .1257 | .1287 | .1317 | .1346 | .1376 |
| 8 | .1405 | .1435 | .1465 | .1495 | .1524 | .1554 |
| 9 | .1584 | .1614 | .1644 | .1673 | .1703 | .1733 |
| 10° | .1763 | .1793 | .1823 | .1853 | .1883 | .1914 |
| 11 | .1944 | .1974 | .2004 | .2035 | .2065 | .2095 |
| 12 | .2126 | .2156 | .2186 | .2217 | .2247 | .2278 |
| 13 | .2309 | .2339 | .2370 | .2401 | .2432 | .2462 |
| 14 | .2493 | .2524 | .2555 | .2586 | .2617 | .2648 |
| 15° | .2679 | .2711 | .2742 | .2773 | .2805 | .2836 |
| 16 | .2867 | .2899 | .2931 | .2962 | .2994 | .3026 |
| 17 | .3057 | .3089 | .3121 | .3153 | .3185 | .3217 |
| 18 | .3249 | .3281 | .3314 | .3346 | .3378 | .3411 |
| 19 | .3443 | .3476 | .3508 | .3541 | .3574 | .3607 |
| 20° | .3640 | .3673 | .3706 | .3739 | .3772 | .3805 |
| 21 | .3839 | .3872 | .3906 | .3939 | .3973 | .4006 |
| 22 | .4040 | .4074 | .4108 | .4142 | .4176 | .4210 |
| 23 | .4245 | .4279 | .4314 | .4348 | .4383 | .4417 |
| 24 | .4452 | .4487 | .4522 | .4557 | .4592 | .4628 |
| 25° | .4663 | .4699 | .4734 | .4770 | .4806 | .4841 |
| 26 | .4877 | .4913 | .4950 | .4986 | .5022 | .5059 |
| 27 | .5095 | .5132 | .5169 | .5206 | .5243 | .5280 |
| 28 | .5317 | .5354 | .5392 | .5430 | .5467 | .5505 |
| 29 | .5543 | .5581 | .5619 | .5658 | .5696 | .5735 |
| 30° | .5774 | .5812 | .5851 | .5890 | .5930 | .5969 |
| 31 | .6009 | .6048 | .6088 | .6128 | .6168 | .6208 |
| 32 | .6249 | .6289 | .6330 | .6371 | .6412 | .6453 |
| 33 | .6494 | .6536 | .6577 | .6619 | .6661 | .6703 |
| 34 | .6745 | .6787 | .6830 | .6873 | .6916 | .6959 |
| 35° | .7002 | .7046 | .7089 | .7133 | .7177 | .7221 |
| 36 | .7265 | .7310 | .7355 | .7400 | .7445 | .7490 |
| 37 | .7536 | .7581 | .7627 | .7673 | .7720 | .7766 |
| 38 | .7813 | .7860 | .7907 | .7954 | .8002 | .8050 |
| 39 | .8098 | .8146 | .8195 | .8243 | .8292 | .8342 |
| 40° | .8391 | .8441 | .8491 | .8541 | .8591 | .8642 |
| 41 | .8693 | .8744 | .8796 | .8847 | .8899 | .8952 |
| 42 | .9004 | .9057 | .9110 | .9163 | .9217 | .9271 |
| 43 | .9325 | .9380 | .9435 | .9490 | .9545 | .9601 |
| 44 | .9657 | .9713 | .9770 | .9827 | .9884 | .9942 |
| 45° | 1.0000 | 1.0058 | 1.0117 | 1.0176 | 1.0235 | 1.0295 |

| $x$ | 0′ | 10′ | 20′ | 30′ | 40′ | 50′ |
|---|---|---|---|---|---|---|
| 45° | 1.0000 | 1.0058 | 1.0117 | 1.0176 | 1.0235 | 1.0295 |
| 46 | 1.0355 | 1.0416 | 1.0477 | 1.0538 | 1.0599 | 1.0661 |
| 47 | 1.0724 | 1.0786 | 1.0850 | 1.0913 | 1.0977 | 1.1041 |
| 48 | 1.1106 | 1.1171 | 1.1237 | 1.1303 | 1.1369 | 1.1436 |
| 49 | 1.1504 | 1.1571 | 1.1640 | 1.1708 | 1.1778 | 1.1847 |
| 50° | 1.1918 | 1.1988 | 1.2059 | 1.2131 | 1.2203 | 1.2276 |
| 51 | 1.2349 | 1.2423 | 1.2497 | 1.2572 | 1.2647 | 1.2723 |
| 52 | 1.2799 | 1.2876 | 1.2954 | 1.3032 | 1.3111 | 1.3190 |
| 53 | 1.3270 | 1.3351 | 1.3432 | 1.3514 | 1.3597 | 1.3680 |
| 54 | 1.3764 | 1.3848 | 1.3934 | 1.4019 | 1.4106 | 1.4193 |
| 55° | 1.4281 | 1.4370 | 1.4460 | 1.4550 | 1.4641 | 1.4733 |
| 56 | 1.4826 | 1.4919 | 1.5013 | 1.5108 | 1.5204 | 1.5301 |
| 57 | 1.5399 | 1.5497 | 1.5597 | 1.5697 | 1.5798 | 1.5900 |
| 58 | 1.6003 | 1.6107 | 1.6212 | 1.6319 | 1.6426 | 1.6534 |
| 59 | 1.6643 | 1.6753 | 1.6864 | 1.6977 | 1.7090 | 1.7205 |
| 60° | 1.7321 | 1.7437 | 1.7556 | 1.7675 | 1.7796 | 1.7917 |
| 61 | 1.8040 | 1.8165 | 1.8291 | 1.8418 | 1.8546 | 1.8676 |
| 62 | 1.8807 | 1.8940 | 1.9074 | 1.9210 | 1.9347 | 1.9486 |
| 63 | 1.9626 | 1.9768 | 1.9912 | 2.0057 | 2.0204 | 2.0353 |
| 64 | 2.0503 | 2.0655 | 2.0809 | 2.0965 | 2.1123 | 2.1283 |
| 65° | 2.1445 | 2.1609 | 2.1775 | 2.1943 | 2.2113 | 2.2286 |
| 66 | 2.2460 | 2.2637 | 2.2817 | 2.2998 | 2.3183 | 2.3369 |
| 67 | 2.3559 | 2.3750 | 2.3945 | 2.4142 | 2.4342 | 2.4545 |
| 68 | 2.4751 | 2.4960 | 2.5172 | 2.5386 | 2.5605 | 2.5826 |
| 69 | 2.6051 | 2.6279 | 2.6511 | 2.6746 | 2.6985 | 2.7228 |
| 70° | 2.7475 | 2.7725 | 2.7980 | 2.8239 | 2.8502 | 2.8770 |
| 71 | 2.9042 | 2.9319 | 2.9600 | 2.9887 | 3.0178 | 3.0475 |
| 72 | 3.0777 | 3.1084 | 3.1397 | 3.1716 | 3.2041 | 3.2371 |
| 73 | 3.2709 | 3.3052 | 3.3402 | 3.3759 | 3.4124 | 3.4495 |
| 74 | 3.4874 | 3.5261 | 3.5656 | 3.6059 | 3.6470 | 3.6891 |
| 75° | 3.7321 | 3.7760 | 3.8208 | 3.8667 | 3.9136 | 3.9617 |
| 76 | 4.0108 | 4.0611 | 4.1126 | 4.1653 | 4.2193 | 4.2747 |
| 77 | 4.3315 | 4.3897 | 4.4494 | 4.5107 | 4.5736 | 4.6382 |
| 78 | 4.7046 | 4.7729 | 4.8430 | 4.9152 | 4.9894 | 5.0658 |
| 79 | 5.1446 | 5.2257 | 5.3093 | 5.3955 | 5.4845 | 5.5764 |
| 80° | 5.6713 | 5.7694 | 5.8708 | 5.9758 | 6.0844 | 6.1970 |
| 81 | 6.3138 | 6.4348 | 6.5606 | 6.6912 | 6.8269 | 6.9682 |
| 82 | 7.1154 | 7.2687 | 7.4287 | 7.5958 | 7.7704 | 7.9530 |
| 83 | 8.1443 | 8.3450 | 8.5555 | 8.7769 | 9.0098 | 9.2553 |
| 84 | 9.5144 | 9.7882 | 10.078 | 10.385 | 10.712 | 11.059 |
| 85° | 11.430 | 11.826 | 12.251 | 12.706 | 13.197 | 13.727 |
| 86 | 14.301 | 14.924 | 15.605 | 16.350 | 17.169 | 18.075 |
| 87 | 19.081 | 20.206 | 21.470 | 22.904 | 24.542 | 26.432 |
| 88 | 28.636 | 31.242 | 34.368 | 38.188 | 42.964 | 49.104 |
| 89 | 57.290 | 68.750 | 85.940 | 114.59 | 171.89 | 343.77 |
| 90° | ∞ | | | | | |

**TABLE**

**6**

## Cot $x$ ($x$ in degrees and minutes)

| $x$ | 0′ | 10′ | 20′ | 30′ | 40′ | 50′ |
|---|---|---|---|---|---|---|
| 0° | ∞ | 343.77 | 171.89 | 114.59 | 85.940 | 68.750 |
| 1 | 57.290 | 49.104 | 42.964 | 38.188 | 34.368 | 31.242 |
| 2 | 28.636 | 26.432 | 24.542 | 22.904 | 21.470 | 20.206 |
| 3 | 19.081 | 18.075 | 17.169 | 16.350 | 15.605 | 14.924 |
| 4 | 14.301 | 13.727 | 13.197 | 12.706 | 12.251 | 11.826 |
| 5° | 11.430 | 11.059 | 10.712 | 10.385 | 10.078 | 9.7882 |
| 6 | 9.5144 | 9.2553 | 9.0098 | 8.7769 | 8.5555 | 8.3450 |
| 7 | 8.1443 | 7.9530 | 7.7704 | 7.5958 | 7.4287 | 7.2687 |
| 8 | 7.1154 | 6.9682 | 6.8269 | 6.6912 | 6.5606 | 6.4348 |
| 9 | 6.3138 | 6.1970 | 6.0844 | 5.9758 | 5.8708 | 5.7694 |
| 10° | 5.6713 | 5.5764 | 5.4845 | 5.3955 | 5.3093 | 5.2257 |
| 11 | 5.1446 | 5.0658 | 4.9894 | 4.9152 | 4.8430 | 4.7729 |
| 12 | 4.7046 | 4.6382 | 4.5736 | 4.5107 | 4.4494 | 4.3897 |
| 13 | 4.3315 | 4.2747 | 4.2193 | 4.1653 | 4.1126 | 4.0611 |
| 14 | 4.0108 | 3.9617 | 3.9136 | 3.8667 | 3.8208 | 3.7760 |
| 15° | 3.7321 | 3.6891 | 3.6470 | 3.6059 | 3.5656 | 3.5261 |
| 16 | 3.4874 | 3.4495 | 3.4124 | 3.3759 | 3.3402 | 3.3052 |
| 17 | 3.2709 | 3.2371 | 3.2041 | 3.1716 | 3.1397 | 3.1084 |
| 18 | 3.0777 | 3.0475 | 3.0178 | 2.9887 | 2.9600 | 2.9319 |
| 19 | 2.9042 | 2.8770 | 2.8502 | 2.8239 | 2.7980 | 2.7725 |
| 20° | 2.7475 | 2.7228 | 2.6985 | 2.6746 | 2.6511 | 2.6279 |
| 21 | 2.6051 | 2.5826 | 2.5605 | 2.5386 | 2.5172 | 2.4960 |
| 22 | 2.4751 | 2.4545 | 2.4342 | 2.4142 | 2.3945 | 2.3750 |
| 23 | 2.3559 | 2.3369 | 3.3183 | 2.2998 | 2.2817 | 2.2637 |
| 24 | 2.2460 | 2.2286 | 2.2113 | 2.1943 | 2.1775 | 2.1609 |
| 25° | 2.1445 | 2.1283 | 2.1123 | 2.0965 | 2.0809 | 2.0655 |
| 26 | 2.0503 | 2.0353 | 2.0204 | 2.0057 | 1.9912 | 1.9768 |
| 27 | 1.9626 | 1.9486 | 1.9347 | 1.9210 | 1.9074 | 1.8940 |
| 28 | 1.8807 | 1.8676 | 1.8546 | 1.8418 | 1.8291 | 1.8165 |
| 29 | 1.8040 | 1.7917 | 1.7796 | 1.7675 | 1.7556 | 1.7437 |
| 30° | 1.7321 | 1.7205 | 1.7090 | 1.6977 | 1.6864 | 1.6753 |
| 31 | 1.6643 | 1.6534 | 1.6426 | 1.6319 | 1.6212 | 1.6107 |
| 32 | 1.6003 | 1.5900 | 1.5798 | 1.5697 | 1.5597 | 1.5497 |
| 33 | 1.5399 | 1.5301 | 1.5204 | 1.5108 | 1.5013 | 1.4919 |
| 34 | 1.4826 | 1.4733 | 1.4641 | 1.4550 | 1.4460 | 1.4370 |
| 35° | 1.4281 | 1.4193 | 1.4106 | 1.4019 | 1.3934 | 1.3848 |
| 36 | 1.3764 | 1.3680 | 1.3597 | 1.3514 | 1.3432 | 1.3351 |
| 37 | 1.3270 | 1.3190 | 1.3111 | 1.3032 | 1.2954 | 1.2876 |
| 38 | 1.2799 | 1.2723 | 1.2647 | 1.2572 | 1.2497 | 1.2423 |
| 39 | 1.2349 | 1.2276 | 1.2203 | 1.2131 | 1.2059 | 1.1988 |
| 40° | 1.1918 | 1.1847 | 1.1778 | 1.1708 | 1.1640 | 1.1571 |
| 41 | 1.1504 | 1.1436 | 1.1369 | 1.1303 | 1.1237 | 1.1171 |
| 42 | 1.1106 | 1.1041 | 1.0977 | 1.0913 | 1.0850 | 1.0786 |
| 43 | 1.0724 | 1.0661 | 1.0599 | 1.0538 | 1.0477 | 1.0416 |
| 44 | 1.0355 | 1.0295 | 1.0235 | 1.0176 | 1.0117 | 1.0058 |
| 45° | 1.0000 | .9942 | .9884 | .9827 | .9770 | .9713 |

| $x$ | 0′ | 10′ | 20′ | 30′ | 40′ | 50′ |
|---|---|---|---|---|---|---|
| 45° | 1.000 | .9942 | .9884 | .9827 | .9770 | .9713 |
| 46 | .9657 | .9601 | .9545 | .9490 | .9435 | .9380 |
| 47 | .9325 | .9271 | .9217 | .9163 | .9110 | .9057 |
| 48 | .9004 | .8952 | .8899 | .8847 | .8796 | .8744 |
| 49 | .8693 | .8642 | .8591 | .8541 | .8491 | .8441 |
| 50° | .8391 | .8342 | .8292 | .8243 | .8195 | .8146 |
| 51 | .8098 | .8050 | .8002 | .7954 | .7907 | .7860 |
| 52 | .7813 | .7766 | .7720 | .7673 | .7627 | .7581 |
| 53 | .7536 | .7490 | .7445 | .7400 | .7355 | .7310 |
| 54 | .7265 | .7221 | .7177 | .7133 | .7089 | .7046 |
| 55° | .7002 | .6959 | .6916 | .6873 | .6830 | .6787 |
| 56 | .6745 | .6703 | .6661 | .6619 | .6577 | .6536 |
| 57 | .6494 | .6453 | .6412 | .6371 | .6330 | .6289 |
| 58 | .6249 | .6208 | .6168 | .6128 | .6088 | .6048 |
| 59 | .6009 | .5969 | .5930 | .5890 | .5851 | .5812 |
| 60° | .5774 | .5735 | .5696 | .5658 | .5619 | .5581 |
| 61 | .5543 | .5505 | .5467 | .5430 | .5392 | .5354 |
| 62 | .5317 | .5280 | .5243 | .5206 | .5169 | .5132 |
| 63 | .5095 | .5059 | .5022 | .4986 | .4950 | .4913 |
| 64 | .4877 | .4841 | .4806 | .4770 | .4734 | .4699 |
| 65° | .4663 | .4628 | .4592 | .4557 | .4522 | .4487 |
| 66 | .4452 | .4417 | .4383 | .4348 | .4314 | .4279 |
| 67 | .4245 | .4210 | .4176 | .4142 | .4108 | .4074 |
| 68 | .4040 | .4006 | .3973 | .3939 | .3906 | .3872 |
| 69 | .3839 | .3805 | .3772 | .3739 | .3706 | .3673 |
| 70° | .3640 | .3607 | .3574 | .3541 | .3508 | .3476 |
| 71 | .3443 | .3411 | .3378 | .3346 | .3314 | .3281 |
| 72 | .3249 | .3217 | .3185 | .3153 | .3121 | .3089 |
| 73 | .3057 | .3026 | .2994 | .2962 | .2931 | .2899 |
| 74 | .2867 | .2836 | .2805 | .2773 | .2742 | .2711 |
| 75° | .2679 | .2648 | .2617 | .2586 | .2555 | .2524 |
| 76 | .2493 | .2462 | .2432 | .2401 | .2370 | .2339 |
| 77 | .2309 | .2278 | .2247 | .2217 | .2186 | .2156 |
| 78 | .2126 | .2095 | .2065 | .2035 | .2004 | .1974 |
| 79 | .1944 | .1914 | .1883 | .1853 | .1823 | .1793 |
| 80° | .1763 | .1733 | .1703 | .1673 | .1644 | .1614 |
| 81 | .1584 | .1554 | .1524 | .1495 | .1465 | .1435 |
| 82 | .1405 | .1376 | .1346 | .1317 | .1287 | .1257 |
| 83 | .1228 | .1198 | .1169 | .1139 | .1110 | .1080 |
| 84 | .1051 | .1022 | .0992 | .0963 | .0934 | .0904 |
| 85° | .0875 | .0846 | .0816 | .0787 | .0758 | .0729 |
| 86 | .0699 | .0670 | .0641 | .0612 | .0582 | .0553 |
| 87 | .0524 | .0495 | .0466 | .0437 | .0407 | .0378 |
| 88 | .0349 | .0320 | .0291 | .0262 | .0233 | .0204 |
| 89 | .0175 | .0145 | .0116 | .0087 | .0058 | .0029 |
| 90° | .0000 | | | | | |

## TABLE 7

## Sec $x$ ($x$ in degrees and minutes)

| $x$ | 0′ | 10′ | 20′ | 30′ | 40′ | 50′ |
|---|---|---|---|---|---|---|
| 0° | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 1 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.001 |
| 2 | 1.001 | 1.001 | 1.001 | 1.001 | 1.001 | 1.001 |
| 3 | 1.001 | 1.002 | 1.002 | 1.002 | 1.002 | 1.002 |
| 4 | 1.002 | 1.003 | 1.003 | 1.003 | 1.003 | 1.004 |
| 5° | 1.004 | 1.004 | 1.004 | 1.005 | 1.005 | 1.005 |
| 6 | 1.006 | 1.006 | 1.006 | 1.006 | 1.007 | 1.007 |
| 7 | 1.008 | 1.008 | 1.008 | 1.009 | 1.009 | 1.009 |
| 8 | 1.010 | 1.010 | 1.011 | 1.011 | 1.012 | 1.012 |
| 9 | 1.012 | 1.013 | 1.013 | 1.014 | 1.014 | 1.015 |
| 10° | 1.015 | 1.016 | 1.016 | 1.017 | 1.018 | 1.018 |
| 11 | 1.019 | 1.019 | 1.020 | 1.020 | 1.021 | 1.022 |
| 12 | 1.022 | 1.023 | 1.024 | 1.024 | 1.025 | 1.026 |
| 13 | 1.026 | 1.027 | 1.028 | 1.028 | 1.029 | 1.030 |
| 14 | 1.031 | 1.031 | 1.032 | 1.033 | 1.034 | 1.034 |
| 15° | 1.035 | 1.036 | 1.037 | 1.038 | 1.039 | 1.039 |
| 16 | 1.040 | 1.041 | 1.042 | 1.043 | 1.044 | 1.045 |
| 17 | 1.046 | 1.047 | 1.048 | 1.048 | 1.049 | 1.050 |
| 18 | 1.051 | 1.052 | 1.053 | 1.054 | 1.056 | 1.057 |
| 19 | 1.058 | 1.059 | 1.060 | 1.061 | 1.062 | 1.063 |
| 20° | 1.064 | 1.065 | 1.066 | 1.068 | 1.069 | 1.070 |
| 21 | 1.071 | 1.072 | 1.074 | 1.075 | 1.076 | 1.077 |
| 22 | 1.079 | 1.080 | 1.081 | 1.082 | 1.084 | 1.085 |
| 23 | 1.086 | 1.088 | 1.089 | 1.090 | 1.092 | 1.093 |
| 24 | 1.095 | 1.096 | 1.097 | 1.099 | 1.100 | 1.102 |
| 25° | 1.103 | 1.105 | 1.106 | 1.108 | 1.109 | 1.111 |
| 26 | 1.113 | 1.114 | 1.116 | 1.117 | 1.119 | 1.121 |
| 27 | 1.122 | 1.124 | 1.126 | 1.127 | 1.129 | 1.131 |
| 28 | 1.133 | 1.134 | 1.136 | 1.138 | 1.140 | 1.142 |
| 29 | 1.143 | 1.145 | 1.147 | 1.149 | 1.151 | 1.153 |
| 30° | 1.155 | 1.157 | 1.159 | 1.161 | 1.163 | 1.165 |
| 31 | 1.167 | 1.169 | 1.171 | 1.173 | 1.175 | 1.177 |
| 32 | 1.179 | 1.181 | 1.184 | 1.186 | 1.188 | 1.190 |
| 33 | 1.192 | 1.195 | 1.197 | 1.199 | 1.202 | 1.204 |
| 34 | 1.206 | 1.209 | 1.211 | 1.213 | 1.216 | 1.218 |
| 35° | 1.221 | 1.223 | 1.226 | 1.228 | 1.231 | 1.233 |
| 36 | 1.236 | 1.239 | 1.241 | 1.244 | 1.247 | 1.249 |
| 37 | 1.252 | 1.255 | 1.258 | 1.260 | 1.263 | 1.266 |
| 38 | 1.269 | 1.272 | 1.275 | 1.278 | 1.281 | 1.284 |
| 39 | 1.287 | 1.290 | 1.293 | 1.296 | 1.299 | 1.302 |
| 40° | 1.305 | 1.309 | 1.312 | 1.315 | 1.318 | 1.322 |
| 41 | 1.325 | 1.328 | 1.332 | 1.335 | 1.339 | 1.342 |
| 42 | 1.346 | 1.349 | 1.353 | 1.356 | 1.360 | 1.364 |
| 43 | 1.367 | 1.371 | 1.375 | 1.379 | 1.382 | 1.386 |
| 44 | 1.390 | 1.394 | 1.398 | 1.402 | 1.406 | 1.410 |
| 45° | 1.414 | 1.418 | 1.423 | 1.427 | 1.431 | 1.435 |

| $x$ | 0′ | 10′ | 20′ | 30′ | 40′ | 50′ |
|---|---|---|---|---|---|---|
| 45° | 1.414 | 1.418 | 1.423 | 1.427 | 1.431 | 1.435 |
| 46 | 1.440 | 1.444 | 1.448 | 1.453 | 1.457 | 1.462 |
| 47 | 1.466 | 1.471 | 1.476 | 1.480 | 1.485 | 1.490 |
| 48 | 1.494 | 1.499 | 1.504 | 1.509 | 1.514 | 1.519 |
| 49 | 1.524 | 1.529 | 1.535 | 1.540 | 1.545 | 1.550 |
| 50° | 1.556 | 1.561 | 1.567 | 1.572 | 1.578 | 1.583 |
| 51 | 1.589 | 1.595 | 1.601 | 1.606 | 1.612 | 1.618 |
| 52 | 1.624 | 1.630 | 1.636 | 1.643 | 1.649 | 1.655 |
| 53 | 1.662 | 1.668 | 1.675 | 1.681 | 1.688 | 1.695 |
| 54 | 1.701 | 1.708 | 1.715 | 1.722 | 1.729 | 1.736 |
| 55° | 1.743 | 1.751 | 1.758 | 1.766 | 1.773 | 1.781 |
| 56 | 1.788 | 1.796 | 1.804 | 1.812 | 1.820 | 1.828 |
| 57 | 1.836 | 1.844 | 1.853 | 1.861 | 1.870 | 1.878 |
| 58 | 1.887 | 1.896 | 1.905 | 1.914 | 1.923 | 1.932 |
| 59 | 1.942 | 1.951 | 1.961 | 1.970 | 1.980 | 1.990 |
| 60° | 2.000 | 2.010 | 2.020 | 2.031 | 2.041 | 2.052 |
| 61 | 2.063 | 2.074 | 2.085 | 2.096 | 2.107 | 2.118 |
| 62 | 2.130 | 2.142 | 2.154 | 2.166 | 2.178 | 2.190 |
| 63 | 2.203 | 2.215 | 2.228 | 2.241 | 2.254 | 2.268 |
| 64 | 2.281 | 2.295 | 2.309 | 2.323 | 2.337 | 2.352 |
| 65° | 2.366 | 2.381 | 2.396 | 2.411 | 2.427 | 2.443 |
| 66 | 2.459 | 2.475 | 2.491 | 2.508 | 2.525 | 2.542 |
| 67 | 2.559 | 2.577 | 2.595 | 2.613 | 2.632 | 2.650 |
| 68 | 2.669 | 2.689 | 2.709 | 2.729 | 2.749 | 2.769 |
| 69 | 2.790 | 2.812 | 2.833 | 2.855 | 2.878 | 2.901 |
| 70° | 2.924 | 2.947 | 2.971 | 2.996 | 3.021 | 3.046 |
| 71 | 3.072 | 3.098 | 3.124 | 3.152 | 3.179 | 3.207 |
| 72 | 3.236 | 3.265 | 3.295 | 3.326 | 3.357 | 3.388 |
| 73 | 3.420 | 3.453 | 3.487 | 3.521 | 3.556 | 3.592 |
| 74 | 3.628 | 3.665 | 3.703 | 3.742 | 3.782 | 3.822 |
| 75° | 3.864 | 3.906 | 3.950 | 3.994 | 4.039 | 4.086 |
| 76 | 4.134 | 4.182 | 4.232 | 4.284 | 4.336 | 4.390 |
| 77 | 4.445 | 4.502 | 4.560 | 4.620 | 4.682 | 4.745 |
| 78 | 4.810 | 4.876 | 4.945 | 5.016 | 5.089 | 5.164 |
| 79 | 5.241 | 5.320 | 5.403 | 5.487 | 5.575 | 5.665 |
| 80° | 5.759 | 5.855 | 5.955 | 6.059 | 6.166 | 6.277 |
| 81 | 6.392 | 6.512 | 6.636 | 6.765 | 6.900 | 7.040 |
| 82 | 7.185 | 7.337 | 7.496 | 7.661 | 7.834 | 8.016 |
| 83 | 8.206 | 8.405 | 8.614 | 8.834 | 9.065 | 9.309 |
| 84 | 9.567 | 9.839 | 10.13 | 10.43 | 10.76 | 11.10 |
| 85° | 11.47 | 11.87 | 12.29 | 12.75 | 13.23 | 13.76 |
| 86 | 14.34 | 14.96 | 15.64 | 16.38 | 17.20 | 18.10 |
| 87 | 19.11 | 20.23 | 21.49 | 22.93 | 24.56 | 26.45 |
| 88 | 28.65 | 31.26 | 34.38 | 38.20 | 42.98 | 49.11 |
| 89 | 57.30 | 68.76 | 85.95 | 114.6 | 171.9 | 343.8 |
| 90° | ∞ | | | | | |

# TABLE 8

# Csc $x$ ($x$ in degrees and minutes)

| $x$ | 0′ | 10′ | 20′ | 30′ | 40′ | 50′ |
|---|---|---|---|---|---|---|
| 0° | ∞ | 343.8 | 171.9 | 114.6 | 85.95 | 68.76 |
| 1 | 57.30 | 49.11 | 42.98 | 38.20 | 34.38 | 31.26 |
| 2 | 28.65 | 26.45 | 24.56 | 22.93 | 21.49 | 20.23 |
| 3 | 19.11 | 18.10 | 17.20 | 16.38 | 15.64 | 14.96 |
| 4 | 14.34 | 13.76 | 13.23 | 12.75 | 12.29 | 11.87 |
| 5° | 11.47 | 11.10 | 10.76 | 10.43 | 10.13 | 9.839 |
| 6 | 9.567 | 9.309 | 9.065 | 8.834 | 8.614 | 8.405 |
| 7 | 8.206 | 8.016 | 7.834 | 7.661 | 7.496 | 7.337 |
| 8 | 7.185 | 7.040 | 6.900 | 6.765 | 6.636 | 6.512 |
| 9 | 6.392 | 6.277 | 6.166 | 6.059 | 5.955 | 5.855 |
| 10° | 5.759 | 5.665 | 5.575 | 5.487 | 5.403 | 5.320 |
| 11 | 5.241 | 5.164 | 5.089 | 5.016 | 4.945 | 4.876 |
| 12 | 4.810 | 4.745 | 4.682 | 4.620 | 4.560 | 4.502 |
| 13 | 4.445 | 4.390 | 4.336 | 4.284 | 4.232 | 4.182 |
| 14 | 4.134 | 4.086 | 4.039 | 3.994 | 3.950 | 3.906 |
| 15° | 3.864 | 3.822 | 3.782 | 3.742 | 3.703 | 3.665 |
| 16 | 3.628 | 3.592 | 3.556 | 3.521 | 3.487 | 3.453 |
| 17 | 3.420 | 3.388 | 3.357 | 3.326 | 3.295 | 3.265 |
| 18 | 3.236 | 3.207 | 3.179 | 3.152 | 3.124 | 3.098 |
| 19 | 3.072 | 3.046 | 3.021 | 2.996 | 2.971 | 2.947 |
| 20° | 2.924 | 2.901 | 2.878 | 2.855 | 2.833 | 2.812 |
| 21 | 2.790 | 2.769 | 2.749 | 2.729 | 2.709 | 2.689 |
| 22 | 2.669 | 2.650 | 2.632 | 2.613 | 2.595 | 2.577 |
| 23 | 2.559 | 2.542 | 2.525 | 2.508 | 2.491 | 2.475 |
| 24 | 2.459 | 2.443 | 2.427 | 2.411 | 2.396 | 2.381 |
| 25° | 2.366 | 2.352 | 2.337 | 2.323 | 2.309 | 2.295 |
| 26 | 2.281 | 2.268 | 2.254 | 2.241 | 2.228 | 2.215 |
| 27 | 2.203 | 2.190 | 2.178 | 2.166 | 2.154 | 2.142 |
| 28 | 2.130 | 2.118 | 2.107 | 2.096 | 2.085 | 2.074 |
| 29 | 2.063 | 2.052 | 2.041 | 2.031 | 2.020 | 2.010 |
| 30° | 2.000 | 1.990 | 1.980 | 1.970 | 1.961 | 1.951 |
| 31 | 1.942 | 1.932 | 1.923 | 1.914 | 1.905 | 1.896 |
| 32 | 1.887 | 1.878 | 1.870 | 1.861 | 1.853 | 1.844 |
| 33 | 1.836 | 1.828 | 1.820 | 1.812 | 1.804 | 1.796 |
| 34 | 1.788 | 1.781 | 1.773 | 1.766 | 1.758 | 1.751 |
| 35° | 1.743 | 1.736 | 1.729 | 1.722 | 1.715 | 1.708 |
| 36 | 1.701 | 1.695 | 1.688 | 1.681 | 1.675 | 1.668 |
| 37 | 1.662 | 1.655 | 1.649 | 1.643 | 1.636 | 1.630 |
| 38 | 1.624 | 1.618 | 1.612 | 1.606 | 1.601 | 1.595 |
| 39 | 1.589 | 1.583 | 1.578 | 1.572 | 1.567 | 1.561 |
| 40° | 1.556 | 1.550 | 1.545 | 1.540 | 1.535 | 1.529 |
| 41 | 1.524 | 1.519 | 1.514 | 1.509 | 1.504 | 1.499 |
| 42 | 1.494 | 1.490 | 1.485 | 1.480 | 1.476 | 1.471 |
| 43 | 1.466 | 1.462 | 1.457 | 1.453 | 1.448 | 1.444 |
| 44 | 1.440 | 1.435 | 1.431 | 1.427 | 1.423 | 1.418 |
| 45° | 1.414 | 1.410 | 1.406 | 1.402 | 1.398 | 1.394 |

| $x$ | 0′ | 10′ | 20′ | 30′ | 40′ | 50′ |
|---|---|---|---|---|---|---|
| 45° | 1.414 | 1.410 | 1.406 | 1.402 | 1.398 | 1.394 |
| 46 | 1.390 | 1.386 | 1.382 | 1.379 | 1.375 | 1.371 |
| 47 | 1.367 | 1.364 | 1.360 | 1.356 | 1.353 | 1.349 |
| 48 | 1.346 | 1.342 | 1.339 | 1.335 | 1.332 | 1.328 |
| 49 | 1.325 | 1.322 | 1.318 | 1.315 | 1.312 | 1.309 |
| 50° | 1.305 | 1.302 | 1.299 | 1.296 | 1.293 | 1.290 |
| 51 | 1.287 | 1.284 | 1.281 | 1.278 | 1.275 | 1.272 |
| 52 | 1.269 | 1.266 | 1.263 | 1.260 | 1.258 | 1.255 |
| 53 | 1.252 | 1.249 | 1.247 | 1.244 | 1.241 | 1.239 |
| 54 | 1.236 | 1.233 | 1.231 | 1.228 | 1.226 | 1.223 |
| 55° | 1.221 | 1.218 | 1.216 | 1.213 | 1.211 | 1.209 |
| 56 | 1.206 | 1.204 | 1.202 | 1.199 | 1.197 | 1.195 |
| 57 | 1.192 | 1.190 | 1.188 | 1.186 | 1.184 | 1.181 |
| 58 | 1.179 | 1.177 | 1.175 | 1.173 | 1.171 | 1.169 |
| 59 | 1.167 | 1.165 | 1.163 | 1.161 | 1.159 | 1.157 |
| 60° | 1.155 | 1.153 | 1.151 | 1.149 | 1.147 | 1.145 |
| 61 | 1.143 | 1.142 | 1.140 | 1.138 | 1.136 | 1.134 |
| 62 | 1.133 | 1.131 | 1.129 | 1.127 | 1.126 | 1.124 |
| 63 | 1.122 | 1.121 | 1.119 | 1.117 | 1.116 | 1.114 |
| 64 | 1.113 | 1.111 | 1.109 | 1.108 | 1.106 | 1.105 |
| 65° | 1.103 | 1.102 | 1.100 | 1.099 | 1.097 | 1.096 |
| 66 | 1.095 | 1.093 | 1.092 | 1.090 | 1.089 | 1.088 |
| 67 | 1.086 | 1.085 | 1.084 | 1.082 | 1.081 | 1.080 |
| 68 | 1.079 | 1.077 | 1.076 | 1.075 | 1.074 | 1.072 |
| 69 | 1.071 | 1.070 | 1.069 | 1.068 | 1.066 | 1.065 |
| 70° | 1.064 | 1.063 | 1.062 | 1.061 | 1.060 | 1.059 |
| 71 | 1.058 | 1.057 | 1.056 | 1.054 | 1.053 | 1.052 |
| 72 | 1.051 | 1.050 | 1.049 | 1.048 | 1.048 | 1.047 |
| 73 | 1.046 | 1.045 | 1.044 | 1.043 | 1.042 | 1.041 |
| 74 | 1.040 | 1.039 | 1.039 | 1.038 | 1.037 | 1.036 |
| 75° | 1.035 | 1.034 | 1.034 | 1.033 | 1.032 | 1.031 |
| 76 | 1.031 | 1.030 | 1.029 | 1.028 | 1.028 | 1.027 |
| 77 | 1.026 | 1.026 | 1.025 | 1.024 | 1.024 | 1.023 |
| 78 | 1.022 | 1.022 | 1.021 | 1.020 | 1.020 | 1.019 |
| 79 | 1.019 | 1.018 | 1.018 | 1.017 | 1.016 | 1.016 |
| 80° | 1.015 | 1.015 | 1.014 | 1.014 | 1.013 | 1.013 |
| 81 | 1.012 | 1.012 | 1.012 | 1.011 | 1.011 | 1.010 |
| 82 | 1.010 | 1.009 | 1.009 | 1.009 | 1.008 | 1.008 |
| 83 | 1.008 | 1.007 | 1.007 | 1.006 | 1.006 | 1.006 |
| 84 | 1.006 | 1.005 | 1.005 | 1.005 | 1.004 | 1.004 |
| 85° | 1.004 | 1.004 | 1.003 | 1.003 | 1.003 | 1.003 |
| 86 | 1.002 | 1.002 | 1.002 | 1.002 | 1.002 | 1.002 |
| 87 | 1.001 | 1.001 | 1.001 | 1.001 | 1.001 | 1.001 |
| 88 | 1.001 | 1.001 | 1.000 | 1.000 | 1.000 | 1.000 |
| 89 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 90° | 1.000 |  |  |  |  |  |

**TABLE**
**9**

## NATURAL TRIGONOMETRIC FUNCTIONS (in radians)

| $x$ | Sin $x$ | Cos $x$ | Tan $x$ | Cot $x$ | Sec $x$ | Csc $x$ |
|---|---|---|---|---|---|---|
| .00 | .00000 | 1.00000 | .00000 | ∞ | 1.00000 | ∞ |
| .01 | .01000 | .99995 | .01000 | 99.9967 | 1.00005 | 100.0017 |
| .02 | .02000 | .99980 | .02000 | 49.9933 | 1.00020 | 50.0033 |
| .03 | .03000 | .99955 | .03001 | 33.3233 | 1.00045 | 33.3383 |
| .04 | .03999 | .99920 | .04002 | 24.9867 | 1.00080 | 25.0067 |
| .05 | .04998 | .99875 | .05004 | 19.9833 | 1.00125 | 20.0083 |
| .06 | .05996 | .99820 | .06007 | 16.6467 | 1.00180 | 16.6767 |
| .07 | .06994 | .99755 | .07011 | 14.2624 | 1.00246 | 14.2974 |
| .08 | .07991 | .99680 | .08017 | 12.4733 | 1.00321 | 12.5133 |
| .09 | .08988 | .99595 | .09024 | 11.0811 | 1.00406 | 11.1261 |
| .10 | .09983 | .99500 | .10033 | 9.9666 | 1.00502 | 10.0167 |
| .11 | .10978 | .99396 | .11045 | 9.0542 | 1.00608 | 9.1093 |
| .12 | .11971 | .99281 | .12058 | 8.2933 | 1.00724 | 8.3534 |
| .13 | .12963 | .99156 | .13074 | 7.6489 | 1.00851 | 7.7140 |
| .14 | .13954 | .99022 | .14092 | 7.0961 | 1.00988 | 7.1662 |
| .15 | .14944 | .98877 | .15114 | 6.6166 | 1.01136 | 6.6917 |
| .16 | .15932 | .98723 | .16138 | 6.1966 | 1.01294 | 6.2767 |
| .17 | .16918 | .98558 | .17166 | 5.8256 | 1.01463 | 5.9108 |
| .18 | .17903 | .98384 | .18197 | 5.4954 | 1.01642 | 5.5857 |
| .19 | .18886 | .98200 | .19232 | 5.1997 | 1.01833 | 5.2950 |
| .20 | .19867 | .98007 | .20271 | 4.9332 | 1.02034 | 5.0335 |
| .21 | .20846 | .97803 | .21314 | 4.6917 | 1.02246 | 4.7971 |
| .22 | .21823 | .97590 | .22362 | 4.4719 | 1.02470 | 4.5823 |
| .23 | .22798 | .97367 | .23414 | 4.2709 | 1.02705 | 4.3864 |
| .24 | .23770 | .97134 | .24472 | 4.0864 | 1.02951 | 4.2069 |
| .25 | .24740 | .96891 | .25534 | 3.9163 | 1.03209 | 4.0420 |
| .26 | .25708 | .96639 | .26602 | 3.7591 | 1.03478 | 3.8898 |
| .27 | .26673 | .96377 | .27676 | 3.6133 | 1.03759 | 3.7491 |
| .28 | .27636 | .96106 | .28755 | 3.4776 | 1.04052 | 3.6185 |
| .29 | .28595 | .95824 | .29841 | 3.3511 | 1.04358 | 3.4971 |
| .30 | .29552 | .95534 | .30934 | 3.2327 | 1.04675 | 3.3839 |
| .31 | .30506 | .95233 | .32033 | 3.1218 | 1.05005 | 3.2781 |
| .32 | .31457 | .94924 | .33139 | 3.0176 | 1.05348 | 3.1790 |
| .33 | .32404 | .94604 | .34252 | 2.9195 | 1.05704 | 3.0860 |
| .34 | .33349 | .94275 | .35374 | 2.8270 | 1.06072 | 2.9986 |
| .35 | .34290 | .93937 | .36503 | 2.7395 | 1.06454 | 2.9163 |
| .36 | .35227 | .93590 | .37640 | 2.6567 | 1.06849 | 2.8387 |
| .37 | .36162 | .93233 | .38786 | 2.5782 | 1.07258 | 2.7654 |
| .38 | .37092 | .92866 | .39941 | 2.5037 | 1.07682 | 2.6960 |
| .39 | .38019 | .92491 | .41105 | 2.4328 | 1.08119 | 2.6303 |
| .40 | .38942 | .92106 | .42279 | 2.3652 | 1.08570 | 2.5679 |

**Table 9**
**(continued)**

## NATURAL TRIGONOMETRIC FUNCTIONS (in radians)

| $x$ | Sin $x$ | Cos $x$ | Tan $x$ | Cot $x$ | Sec $x$ | Csc $x$ |
|---|---|---|---|---|---|---|
| .40 | .38942 | .92106 | .42279 | 2.3652 | 1.0857 | 2.5679 |
| .41 | .39861 | .91712 | .43463 | 2.3008 | 1.0904 | 2.5087 |
| .42 | .40776 | .91309 | .44657 | 2.2393 | 1.0952 | 2.4524 |
| .43 | .41687 | .90897 | .45862 | 2.1804 | 1.1002 | 2.3988 |
| .44 | .42594 | .90475 | .47078 | 2.1241 | 1.1053 | 2.3478 |
| .45 | .43497 | .90045 | .48306 | 2.0702 | 1.1106 | 2.2990 |
| .46 | .44395 | .89605 | .49545 | 2.0184 | 1.1160 | 2.2525 |
| .47 | .45289 | .89157 | .50797 | 1.9686 | 1.1216 | 2.2081 |
| .48 | .46178 | .88699 | .52061 | 1.9208 | 1.1274 | 2.1655 |
| .49 | .47063 | .88233 | .53339 | 1.8748 | 1.1334 | 2.1248 |
| .50 | .47943 | .87758 | .54630 | 1.8305 | 1.1395 | 2.0858 |
| .51 | .48818 | .87274 | .55936 | 1.7878 | 1.1458 | 2.0484 |
| .52 | .49688 | .86782 | .57256 | 1.7465 | 1.1523 | 2.0126 |
| .53 | .50553 | .86281 | .58592 | 1.7067 | 1.1590 | 1.9781 |
| .54 | .51414 | .85771 | .59943 | 1.6683 | 1.1659 | 1.9450 |
| .55 | .52269 | .85252 | .61311 | 1.6310 | 1.1730 | 1.9132 |
| .56 | .53119 | .84726 | .62695 | 1.5950 | 1.1803 | 1.8826 |
| .57 | .53963 | .84190 | .64097 | 1.5601 | 1.1878 | 1.8531 |
| .58 | .54802 | .83646 | .65517 | 1.5263 | 1.1955 | 1.8247 |
| .59 | .55636 | .83094 | .66956 | 1.4935 | 1.2035 | 1.7974 |
| .60 | .56464 | .82534 | .68414 | 1.4617 | 1.2116 | 1.7710 |
| .61 | .57287 | .81965 | .69892 | 1.4308 | 1.2200 | 1.7456 |
| .62 | .58104 | .81388 | .71391 | 1.4007 | 1.2287 | 1.7211 |
| .63 | .58914 | .80803 | .72911 | 1.3715 | 1.2376 | 1.6974 |
| .64 | .59720 | .80210 | .74454 | 1.3431 | 1.2467 | 1.6745 |
| .65 | .60519 | .79608 | .76020 | 1.3154 | 1.2561 | 1.6524 |
| .66 | .61312 | .78999 | .77610 | 1.2885 | 1.2658 | 1.6310 |
| .67 | .62099 | .78382 | .79225 | 1.2622 | 1.2758 | 1.6103 |
| .68 | .62879 | .77757 | .80866 | 1.2366 | 1.2861 | 1.5903 |
| .69 | .63654 | .77125 | .82534 | 1.2116 | 1.2966 | 1.5710 |
| .70 | .64422 | .76484 | .84229 | 1.1872 | 1.3075 | 1.5523 |
| .71 | .65183 | .75836 | .85953 | 1.1634 | 1.3186 | 1.5341 |
| .72 | .65938 | .75181 | .87707 | 1.1402 | 1.3301 | 1.5166 |
| .73 | .66687 | .74517 | .89492 | 1.1174 | 1.3420 | 1.4995 |
| .74 | .67429 | .73847 | .91309 | 1.0952 | 1.3542 | 1.4830 |
| .75 | .68164 | .73169 | .93160 | 1.0734 | 1.3667 | 1.4671 |
| .76 | .68892 | .72484 | .95045 | 1.0521 | 1.3796 | 1.4515 |
| .77 | .69614 | .71791 | .96967 | 1.0313 | 1.3929 | 1.4365 |
| .78 | .70328 | .71091 | .98926 | 1.0109 | 1.4066 | 1.4219 |
| .79 | .71035 | .70385 | 1.0092 | .99084 | 1.4208 | 1.4078 |
| .80 | .71736 | .69671 | 1.0296 | .97121 | 1.4353 | 1.3940 |

**Table 9**
**(continued)**

## NATURAL TRIGONOMETRIC FUNCTIONS (in radians)

| $x$ | Sin $x$ | Cos $x$ | Tan $x$ | Cot $x$ | Sec $x$ | Csc $x$ |
|---|---|---|---|---|---|---|
| .80 | .71736 | .69671 | 1.0296 | .97121 | 1.4353 | 1.3940 |
| .81 | .72429 | .68950 | 1.0505 | .95197 | 1.4503 | 1.3807 |
| .82 | .73115 | .68222 | 1.0717 | .93309 | 1.4658 | 1.3677 |
| .83 | .73793 | .67488 | 1.0934 | .91455 | 1.4818 | 1.3551 |
| .84 | .74464 | .66746 | 1.1156 | .89635 | 1.4982 | 1.3429 |
| .85 | .75128 | .65998 | 1.1383 | .87848 | 1.5152 | 1.3311 |
| .86 | .75784 | .65244 | 1.1616 | .86091 | 1.5327 | 1.3195 |
| .87 | .76433 | .64483 | 1.1853 | .84365 | 1.5508 | 1.3083 |
| .88 | .77074 | .63715 | 1.2097 | .82668 | 1.5695 | 1.2975 |
| .89 | .77707 | .62941 | 1.2346 | .80998 | 1.5888 | 1.2869 |
| .90 | .78333 | .62161 | 1.2602 | .79355 | 1.6087 | 1.2766 |
| .91 | .78950 | .61375 | 1.2864 | .77738 | 1.6293 | 1.2666 |
| .92 | .79560 | .60582 | 1.3133 | .76146 | 1.6507 | 1.2569 |
| .93 | .80162 | .59783 | 1.3409 | .74578 | 1.6727 | 1.2475 |
| .94 | .80756 | .58979 | 1.3692 | .73034 | 1.6955 | 1.2383 |
| .95 | .81342 | .58168 | 1.3984 | .71511 | 1.7191 | 1.2294 |
| .96 | .81919 | .57352 | 1.4284 | .70010 | 1.7436 | 1.2207 |
| .97 | .82489 | .56530 | 1.4592 | .68531 | 1.7690 | 1.2123 |
| .98 | .83050 | .55702 | 1.4910 | .67071 | 1.7953 | 1.2041 |
| .99 | .83603 | .54869 | 1.5237 | .65631 | 1.8225 | 1.1961 |
| 1.00 | .84147 | .54030 | 1.5574 | .64209 | 1.8508 | 1.1884 |
| 1.01 | .84683 | .53186 | 1.5922 | .62806 | 1.8802 | 1.1809 |
| 1.02 | .85211 | .52337 | 1.6281 | .61420 | 1.9107 | 1.1736 |
| 1.03 | .85730 | .51482 | 1.6652 | .60051 | 1.9424 | 1.1665 |
| 1.04 | .86240 | .50622 | 1.7036 | .58699 | 1.9754 | 1.1595 |
| 1.05 | .86742 | .49757 | 1.7433 | .57362 | 2.0098 | 1.1528 |
| 1.06 | .87236 | .48887 | 1.7844 | .56040 | 2.0455 | 1.1463 |
| 1.07 | .87720 | .48012 | 1.8270 | .54734 | 2.0828 | 1.1400 |
| 1.08 | .88196 | .47133 | 1.8712 | .53441 | 2.1217 | 1.1338 |
| 1.09 | .88663 | .46249 | 1.9171 | .52162 | 2.1622 | 1.1279 |
| 1.10 | .89121 | .45360 | 1.9648 | .50897 | 2.2046 | 1.1221 |
| 1.11 | .89570 | .44466 | 2.0143 | .49644 | 2.2489 | 1.1164 |
| 1.12 | .90010 | .43568 | 2.0660 | .48404 | 2.2952 | 1.1110 |
| 1.13 | .90441 | .42666 | 2.1198 | .47175 | 2.3438 | 1.1057 |
| 1.14 | .90863 | .41759 | 2.1759 | .45959 | 2.3947 | 1.1006 |
| 1.15 | .91276 | .40849 | 2.2345 | .44753 | 2.4481 | 1.0956 |
| 1.16 | .91680 | .39934 | 2.2958 | .43558 | 2.5041 | 1.0907 |
| 1.17 | .92075 | .39015 | 2.3600 | .42373 | 2.5631 | 1.0861 |
| 1.18 | .92461 | .38092 | 2.4273 | .41199 | 2.6252 | 1.0815 |
| 1.19 | .92837 | .37166 | 2.4979 | .40034 | 2.6906 | 1.0772 |
| 1.20 | .93204 | .36236 | 2.5722 | .38878 | 2.7597 | 1.0729 |

**Table 9**
**(continued)**

## NATURAL TRIGONOMETRIC FUNCTIONS (in radians)

| $x$ | Sin $x$ | Cos $x$ | Tan $x$ | Cot $x$ | Sec $x$ | Csc $x$ |
|---|---|---|---|---|---|---|
| 1.20 | .93204 | .36236 | 2.5722 | .38878 | 2.7597 | 1.07292 |
| 1.21 | .93562 | .35302 | 2.6503 | .37731 | 2.8327 | 1.06881 |
| 1.22 | .93910 | .34365 | 2.7328 | .36593 | 2.9100 | 1.06485 |
| 1.23 | .94249 | .33424 | 2.8198 | .35463 | 2.9919 | 1.06102 |
| 1.24 | .94578 | .32480 | 2.9119 | .34341 | 3.0789 | 1.05732 |
| 1.25 | .94898 | .31532 | 3.0096 | .33227 | 3.1714 | 1.05376 |
| 1.26 | .95209 | .30582 | 3.1133 | .32121 | 3.2699 | 1.05032 |
| 1.27 | .95510 | .29628 | 3.2236 | .31021 | 3.3752 | 1.04701 |
| 1.28 | .95802 | .28672 | 3.3414 | .29928 | 3.4878 | 1.04382 |
| 1.29 | .96084 | .27712 | 3.4672 | .28842 | 3.6085 | 1.04076 |
| 1.30 | .96356 | .26750 | 3.6021 | .27762 | 3.7383 | 1.03782 |
| 1.31 | .96618 | .25785 | 3.7471 | .26687 | 3.8782 | 1.03500 |
| 1.32 | .96872 | .24818 | 3.9033 | .25619 | 4.0294 | 1.03230 |
| 1.33 | .97115 | .23848 | 4.0723 | .24556 | 4.1933 | 1.02971 |
| 1.34 | .97348 | .22875 | 4.2556 | .23498 | 4.3715 | 1.02724 |
| 1.35 | .97572 | .21901 | 4.4552 | .22446 | 4.5661 | 1.02488 |
| 1.36 | .97786 | .20924 | 4.6734 | .21398 | 4.7792 | 1.02264 |
| 1.37 | .97991 | .19945 | 4.9131 | .20354 | 5.0138 | 1.02050 |
| 1.38 | .98185 | .18964 | 5.1774 | .19315 | 5.2731 | 1.01848 |
| 1.39 | .98370 | .17981 | 5.4707 | .18279 | 5.5613 | 1.01657 |
| 1.40 | .98545 | .16997 | 5.7979 | .17248 | 5.8835 | 1.01477 |
| 1.41 | .98710 | .16010 | 6.1654 | .16220 | 6.2459 | 1.01307 |
| 1.42 | .98865 | .15023 | 6.5811 | .15195 | 6.6567 | 1.01148 |
| 1.43 | .99010 | .14033 | 7.0555 | .14173 | 7.1260 | 1.00999 |
| 1.44 | .99146 | .13042 | 7.6018 | .13155 | 7.6673 | 1.00862 |
| 1.45 | .99271 | .12050 | 8.2381 | .12139 | 8.2986 | 1.00734 |
| 1.46 | .99387 | .11057 | 8.9886 | .11125 | 9.0441 | 1.00617 |
| 1.47 | .99492 | .10063 | 9.8874 | .10114 | 9.9378 | 1.00510 |
| 1.48 | .99588 | .09067 | 10.9834 | .09105 | 11.0288 | 1.00414 |
| 1.49 | .99674 | .08071 | 12.3499 | .08097 | 12.3903 | 1.00327 |
| 1.50 | .99749 | .07074 | 14.1014 | .07091 | 14.1368 | 1.00251 |
| 1.51 | .99815 | .06076 | 16.4281 | .06087 | 16.4585 | 1.00185 |
| 1.52 | .99871 | .05077 | 19.6695 | .05084 | 19.6949 | 1.00129 |
| 1.53 | .99917 | .04079 | 24.4984 | .04082 | 24.5188 | 1.00083 |
| 1.54 | .99953 | .03079 | 32.4611 | .03081 | 32.4765 | 1.00047 |
| 1.55 | .99978 | .02079 | 48.0785 | .02080 | 48.0889 | 1.00022 |
| 1.56 | .99994 | .01080 | 92.6205 | .01080 | 92.6259 | 1.00006 |
| 1.57 | 1.00000 | .00080 | 1255.77 | .00080 | 1255.77 | 1.00000 |
| 1.58 | .99996 | −.00920 | −108.649 | −.00920 | −108.654 | 1.00004 |
| 1.59 | .99982 | −.01920 | −52.0670 | −.01921 | −52.0766 | 1.00018 |
| 1.60 | .99957 | −.02920 | −34.2325 | −.02921 | −34.2471 | 1.00043 |

## TABLE 10

## **log sin** $x$ ($x$ in degrees and minutes)

[subtract 10 from each entry]

| $x$ | 0′ | 10′ | 20′ | 30′ | 40′ | 50′ |
|---|---|---|---|---|---|---|
| 0° | – | 7.4637 | 7.7648 | 7.9408 | 8.0658 | 8.1627 |
| 1 | 8.2419 | 8.3088 | 8.3668 | 8.4179 | 8.4637 | 8.5050 |
| 2 | 8.5428 | 8.5776 | 8.6097 | 8.6397 | 8.6677 | 8.6940 |
| 3 | 8.7188 | 8.7423 | 8.7645 | 8.7857 | 8.8059 | 8.8251 |
| 4 | 8.8436 | 8.8613 | 8.8783 | 8.8946 | 8.9104 | 8.9256 |
| 5° | 8.9403 | 8.9545 | 8.9682 | 8.9816 | 8.9945 | 9.0070 |
| 6 | 9.0192 | 9.0311 | 9.0426 | 9.0539 | 9.0648 | 9.0755 |
| 7 | 9.0859 | 9.0961 | 9.1060 | 9.1157 | 9.1252 | 9.1345 |
| 8 | 9.1436 | 9.1525 | 9.1612 | 9.1697 | 9.1781 | 9.1863 |
| 9 | 9.1943 | 9.2022 | 9.2100 | 9.2176 | 9.2251 | 9.2324 |
| 10° | 9.2397 | 9.2468 | 9.2538 | 9.2606 | 9.2674 | 9.2740 |
| 11 | 9.2806 | 9.2870 | 9.2934 | 9.2997 | 9.3058 | 9.3119 |
| 12 | 9.3179 | 9.3238 | 9.3296 | 9.3353 | 9.3410 | 9.3466 |
| 13 | 9.3521 | 9.3575 | 9.3629 | 9.3682 | 9.3734 | 9.3786 |
| 14 | 9.3837 | 9.3887 | 9.3937 | 9.3986 | 9.4035 | 9.4083 |
| 15° | 9.4130 | 9.4177 | 9.4223 | 9.4269 | 9.4314 | 9.4359 |
| 16 | 9.4403 | 9.4447 | 9.4491 | 9.4533 | 9.4576 | 9.4618 |
| 17 | 9.4659 | 9.4700 | 9.4741 | 9.4781 | 9.4821 | 9.4861 |
| 18 | 9.4900 | 9.4939 | 9.4977 | 9.5015 | 9.5052 | 9.5090 |
| 19 | 9.5126 | 9.5163 | 9.5199 | 9.5235 | 9.5270 | 9.5306 |
| 20° | 9.5341 | 9.5375 | 9.5409 | 9.5443 | 9.5477 | 9.5510 |
| 21 | 9.5543 | 9.5576 | 9.5609 | 9.5641 | 9.5673 | 9.5704 |
| 22 | 9.5736 | 9.5767 | 9.5798 | 9.5828 | 9.5859 | 9.5889 |
| 23 | 9.5919 | 9.5948 | 9.5978 | 9.6007 | 9.6036 | 9.6065 |
| 24 | 9.6093 | 9.6121 | 9.6149 | 9.6177 | 9.6205 | 9.6232 |
| 25° | 9.6259 | 9.6286 | 9.6313 | 9.6340 | 9.6366 | 9.6392 |
| 26 | 9.6418 | 9.6444 | 9.6470 | 9.6495 | 9.6521 | 9.6546 |
| 27 | 9.6570 | 9.6595 | 9.6620 | 9.6644 | 9.6668 | 9.6692 |
| 28 | 9.6716 | 9.6740 | 9.6763 | 9.6787 | 9.6810 | 9.6833 |
| 29 | 9.6856 | 9.6878 | 9.6901 | 9.6923 | 9.6946 | 9.6968 |
| 30° | 9.6990 | 9.7012 | 9.7033 | 9.7055 | 9.7076 | 9.7097 |
| 31 | 9.7118 | 9.7139 | 9.7160 | 9.7181 | 9.7201 | 9.7222 |
| 32 | 9.7242 | 9.7262 | 9.7282 | 9.7302 | 9.7322 | 9.7342 |
| 33 | 9.7361 | 9.7380 | 9.7400 | 9.7419 | 9.7438 | 9.7457 |
| 34 | 9.7476 | 9.7494 | 9.7513 | 9.7531 | 9.7550 | 9.7568 |
| 35° | 9.7586 | 9.7604 | 9.7622 | 9.7640 | 9.7657 | 9.7675 |
| 36 | 9.7692 | 9.7710 | 9.7727 | 9.7744 | 9.7761 | 9.7778 |
| 37 | 9.7795 | 9.7811 | 9.7828 | 9.7844 | 9.7861 | 9.7877 |
| 38 | 9.7893 | 9.7910 | 9.7926 | 9.7941 | 9.7957 | 9.7973 |
| 39 | 9.7989 | 9.8004 | 9.8020 | 9.8035 | 9.8050 | 9.8066 |
| 40° | 9.8081 | 9.8096 | 9.8111 | 9.8125 | 9.8140 | 9.8155 |
| 41 | 9.8169 | 9.8184 | 9.8198 | 9.8213 | 9.8227 | 9.8241 |
| 42 | 9.8255 | 9.8269 | 9.8283 | 9.8297 | 9.8311 | 9.8324 |
| 43 | 9.8338 | 9.8351 | 9.8365 | 9.8378 | 9.8391 | 9.8405 |
| 44 | 9.8418 | 9.8431 | 9.8444 | 9.8457 | 9.8469 | 9.8482 |
| 45° | 9.8495 | 9.8507 | 9.8520 | 9.8532 | 9.8545 | 9.8557 |

**Table 10**
**(continued)**

## log sin $x$ ($x$ in degrees and minutes)

[subtract 10 from each entry]

| $x$ | 0′ | 10′ | 20′ | 30′ | 40′ | 50′ |
|---|---|---|---|---|---|---|
| 45° | 9.8495 | 9.8507 | 9.8520 | 9.8532 | 9.8545 | 9.8557 |
| 46 | 9.8569 | 9.8582 | 9.8594 | 9.8606 | 9.8618 | 9.8629 |
| 47 | 9.8641 | 9.8653 | 9.8665 | 9.8676 | 9.8688 | 9.8699 |
| 48 | 9.8711 | 9.8722 | 9.8733 | 9.8745 | 9.8756 | 9.8767 |
| 49 | 9.8778 | 9.8789 | 9.8800 | 9.8810 | 9.8821 | 9.8832 |
| 50° | 9.8843 | 9.8853 | 9.8864 | 9.8874 | 9.8884 | 9.8895 |
| 51 | 9.8905 | 9.8915 | 9.8925 | 9.8935 | 9.8945 | 3.8955 |
| 52 | 9.8965 | 9.8975 | 9.8985 | 9.8995 | 9.9004 | 9.9014 |
| 53 | 9.9023 | 9.9033 | 9.9042 | 9.9052 | 9.9061 | 9.9070 |
| 54 | 9.9080 | 9.9089 | 9.9098 | 9.9107 | 9.9116 | 9.9125 |
| 55° | 9.9134 | 9.9142 | 9.9151 | 9.9160 | 9.9169 | 9.9177 |
| 56 | 9.9186 | 9.9194 | 9.9203 | 9.9211 | 9.9219 | 9.9228 |
| 57 | 9.9236 | 9.9244 | 9.9252 | 9.9260 | 9.9268 | 9.9276 |
| 58 | 9.9284 | 9.9292 | 9.9300 | 9.9308 | 9.9315 | 9.9323 |
| 59 | 9.9331 | 9.9338 | 9.9346 | 9.9353 | 9.9361 | 9.9368 |
| 60° | 9.9375 | 9.9383 | 9.9390 | 9.9397 | 9.9404 | 9.9411 |
| 61 | 9.9418 | 9.9425 | 9.9432 | 9.9439 | 9.9446 | 9.9453 |
| 62 | 9.9459 | 9.9466 | 9.9473 | 9.9479 | 9.9486 | 9.9492 |
| 63 | 9.9499 | 9.9505 | 9.9512 | 9.9518 | 9.9524 | 9.9530 |
| 64 | 9.9537 | 9.9543 | 9.9549 | 9.9555 | 9.9561 | 9.9567 |
| 65° | 9.9573 | 9.9579 | 9.9584 | 9.9590 | 9.9596 | 9.9602 |
| 66 | 9.9607 | 9.9613 | 9.9618 | 9.9624 | 9.9629 | 9.9635 |
| 67 | 9.9640 | 9.9646 | 9.9651 | 9.9656 | 9.9661 | 9.9667 |
| 68 | 9.9672 | 9.9677 | 9.9682 | 9.9687 | 9.9692 | 9.9697 |
| 69 | 9.9702 | 9.9706 | 9.9711 | 9.9716 | 9.9721 | 9.9725 |
| 70° | 9.9730 | 9.9734 | 9.9739 | 9.9743 | 9.9748 | 9.9752 |
| 71 | 9.9757 | 9.9761 | 9.9765 | 9.9770 | 9.9774 | 9.9778 |
| 72 | 9.9782 | 9.9786 | 9.9790 | 9.9794 | 9.9798 | 9.9802 |
| 73 | 9.9806 | 9.9810 | 9.9814 | 9.9817 | 9.9821 | 9.9825 |
| 74 | 9.9828 | 9.9832 | 9.9836 | 9.9839 | 9.9843 | 9.9846 |
| 75° | 9.9849 | 9.9853 | 9.9856 | 9.9859 | 9.9863 | 9.9866 |
| 76 | 9.9869 | 9.9872 | 9.9875 | 9.9878 | 9.9881 | 9.9884 |
| 77 | 9.9887 | 9.9890 | 9.9893 | 9.9896 | 9.9899 | 9.9901 |
| 78 | 9.9904 | 9.9907 | 9.9909 | 9.9912 | 9.9914 | 9.9917 |
| 79 | 9.9919 | 9.9922 | 9.9924 | 9.9927 | 9.9929 | 9.9931 |
| 80° | 9.9934 | 9.9936 | 9.9938 | 9.9940 | 9.9942 | 9.9944 |
| 81 | 9.9946 | 9.9948 | 9.9950 | 9.9952 | 9.9954 | 9.9956 |
| 82 | 9.9958 | 9.9959 | 9.9961 | 9.9963 | 9.9964 | 9.9966 |
| 83 | 9.9968 | 9.9969 | 9.9971 | 9.9972 | 9.9973 | 9.9975 |
| 84 | 9.9976 | 9.9977 | 9.9979 | 9.9980 | 9.9981 | 9.9982 |
| 85° | 9.9983 | 9.9985 | 9.9986 | 9.9987 | 9.9988 | 9.9989 |
| 86 | 9.9989 | 9.9990 | 9.9991 | 9.9992 | 9.9993 | 9.9993 |
| 87 | 9.9994 | 9.9995 | 9.9995 | 9.9996 | 9.9996 | 9.9997 |
| 88 | 9.9997 | 9.9998 | 9.9998 | 9.9999 | 9.9999 | 9.9999 |
| 89 | 9.9999 | 10.0000 | 10.0000 | 10.0000 | 10.0000 | 10.0000 |
| 90° | 10.0000 | | | | | |

## TABLE 11

# log cos $x$ ($x$ in degrees and minutes)

[subtract 10 from each entry]

| $x$ | 0′ | 10′ | 20′ | 30′ | 40′ | 50′ |
|---|---|---|---|---|---|---|
| 0° | 10.0000 | 10.0000 | 10.0000 | 10.0000 | 10.0000 | 10.0000 |
| 1 | 9.9999 | 9.9999 | 9.9999 | 9.9999 | 9.9998 | 9.9998 |
| 2 | 9.9997 | 9.9997 | 9.9996 | 9.9996 | 9.9995 | 9.9995 |
| 3 | 9.9994 | 9.9993 | 9.9993 | 9.9992 | 9.9991 | 9.9990 |
| 4 | 9.9989 | 9.9989 | 9.9988 | 9.9987 | 9.9986 | 9.9985 |
| 5° | 9.9983 | 9.9982 | 9.9981 | 9.9980 | 9.9979 | 9.9977 |
| 6 | 9.9976 | 9.9975 | 9.9973 | 9.9972 | 9.9971 | 9.9969 |
| 7 | 9.9968 | 9.9966 | 9.9964 | 9.9963 | 9.9961 | 9.9959 |
| 8 | 9.9958 | 9.9956 | 9.9954 | 9.9952 | 9.9950 | 9.9948 |
| 9 | 9.9946 | 9.9944 | 9.9942 | 9.9940 | 9.9938 | 9.9936 |
| 10° | 9.9934 | 9.9931 | 9.9929 | 9.9927 | 9.9924 | 9.9922 |
| 11 | 9.9919 | 9.9917 | 9.9914 | 9.9912 | 9.9909 | 9.9907 |
| 12 | 9.9904 | 9.9901 | 9.9899 | 9.9896 | 9.9893 | 9.9890 |
| 13 | 9.9887 | 9.9884 | 9.9881 | 9.9878 | 9.9875 | 9.9872 |
| 14 | 9.9869 | 9.9866 | 9.9863 | 9.9859 | 9.9856 | 9.9853 |
| 15° | 9.9849 | 9.9846 | 9.9843 | 9.9839 | 9.9836 | 9.9832 |
| 16 | 9.9828 | 9.9825 | 9.9821 | 9.9817 | 9.9814 | 9.9810 |
| 17 | 9.9806 | 9.9802 | 9.9798 | 9.9794 | 9.9790 | 9.9786 |
| 18 | 9.9782 | 9.9778 | 9.9774 | 9.9770 | 9.9765 | 9.9761 |
| 19 | 9.9757 | 9.9752 | 9.9748 | 9.9743 | 9.9739 | 9.9734 |
| 20° | 9.9730 | 9.9725 | 9.9721 | 9.9716 | 9.9711 | 9.9706 |
| 21 | 9.9702 | 9.9697 | 9.9692 | 9.9687 | 9.9682 | 9.9677 |
| 22 | 9.9672 | 9.9667 | 9.9661 | 9.9656 | 9.9651 | 9.9646 |
| 23 | 9.9640 | 9.9635 | 9.9629 | 9.9624 | 9.9618 | 9.9613 |
| 24 | 9.9607 | 9.9602 | 9.9596 | 9.9590 | 9.9584 | 9.9579 |
| 25° | 9.9573 | 9.9567 | 9.9561 | 9.9555 | 9.9549 | 9.9543 |
| 26 | 9.9537 | 9.9530 | 9.9524 | 9.9518 | 9.9512 | 9.9505 |
| 27 | 9.9499 | 9.9492 | 9.9486 | 9.9479 | 9.9473 | 9.9466 |
| 28 | 9.9459 | 9.9453 | 9.9446 | 9.9439 | 9.9432 | 9.9425 |
| 29 | 9.9418 | 9.9411 | 9.9404 | 9.9397 | 9.9390 | 9.9383 |
| 30° | 9.9375 | 9.9368 | 9.9361 | 9.9353 | 9.9346 | 9.9338 |
| 31 | 9.9331 | 9.9323 | 9.9315 | 9.9308 | 9.9300 | 9.9292 |
| 32 | 9.9284 | 9.9276 | 9.9268 | 9.9260 | 9.9252 | 9.9244 |
| 33 | 9.9236 | 9.9228 | 9.9219 | 9.9211 | 9.9203 | 9.9194 |
| 34 | 9.9186 | 9.9177 | 9.9169 | 9.9160 | 9.9151 | 9.9142 |
| 35° | 9.9134 | 9.9125 | 9.9116 | 9.9107 | 9.9098 | 9.9089 |
| 36 | 9.9080 | 9.9070 | 9.9061 | 9.9052 | 9.9042 | 9.9033 |
| 37 | 9.9023 | 9.9014 | 9.9004 | 9.8995 | 9.8985 | 9.8975 |
| 38 | 9.8965 | 9.8955 | 9.8945 | 9.8935 | 9.8925 | 9.8915 |
| 39 | 9.8905 | 9.8895 | 9.8884 | 9.8874 | 9.8864 | 9.8853 |
| 40° | 9.8843 | 9.8832 | 9.8821 | 9.8810 | 9.8800 | 9.8789 |
| 41 | 9.8778 | 9.8767 | 9.8756 | 9.8745 | 9.8733 | 9.8722 |
| 42 | 9.8711 | 9.8699 | 9.8688 | 9.8676 | 9.8665 | 9.8653 |
| 43 | 9.8641 | 9.8629 | 9.8618 | 9.8606 | 9.8594 | 9.8582 |
| 44 | 9.8569 | 9.8557 | 9.8545 | 9.8532 | 9.8520 | 9.8507 |
| 45° | 9.8495 | 9.8482 | 9.8469 | 9.8457 | 9.8444 | 9.8431 |

**Table 11**
**(continued)**

## log cos $x$ ($x$ in degrees and minutes)

[subtract 10 from each entry]

| $x$ | 0′ | 10′ | 20′ | 30′ | 40′ | 50′ |
|---|---|---|---|---|---|---|
| 45° | 9.8495 | 9.8482 | 9.8469 | 9.8457 | 9.8444 | 9.8431 |
| 46 | 9.8418 | 9.8405 | 9.8391 | 9.8378 | 9.8365 | 9.8351 |
| 47 | 9.8338 | 9.8324 | 9.8311 | 9.8297 | 9.8283 | 9.8269 |
| 48 | 9.8255 | 9.8241 | 9.8227 | 9.8213 | 9.8198 | 9.8184 |
| 49 | 9.8169 | 9.8155 | 9.8140 | 9.8125 | 9.8111 | 9.8096 |
| 50° | 9.8081 | 9.8066 | 9.8050 | 9.8035 | 9.8020 | 9.8004 |
| 51 | 9.7989 | 9.7973 | 9.7957 | 9.7941 | 9.7926 | 9.7910 |
| 52 | 9.7893 | 9.7877 | 9.7861 | 9.7844 | 9.7828 | 9.7811 |
| 53 | 9.7795 | 9.7778 | 9.7761 | 9.7744 | 9.7727 | 9.7710 |
| 54 | 9.7692 | 9.7675 | 9.7657 | 9.7640 | 9.7622 | 9.7604 |
| 55° | 9.7586 | 9.7568 | 9.7550 | 9.7531 | 9.7513 | 9.7494 |
| 56 | 9.7476 | 9.7457 | 9.7438 | 9.7419 | 9.7400 | 9.7380 |
| 57 | 9.7361 | 9.7342 | 9.7322 | 9.7302 | 9.7282 | 9.7262 |
| 58 | 9.7242 | 9.7222 | 9.7201 | 9.7181 | 9.7160 | 9.7139 |
| 59 | 9.7118 | 9.7097 | 9.7076 | 9.7055 | 9.7033 | 9.7012 |
| 60° | 9.6990 | 9.6968 | 9.6946 | 9.6923 | 9.6901 | 9.6878 |
| 61 | 9.6856 | 9.6833 | 9.6810 | 9.6787 | 9.6763 | 9.6740 |
| 62 | 9.6716 | 9.6692 | 9.6668 | 9.6644 | 9.6620 | 9.6595 |
| 63 | 9.6570 | 9.6546 | 9.6521 | 9.6495 | 9.6470 | 9.6444 |
| 64 | 9.6418 | 9.6392 | 9.6366 | 9.6340 | 9.6313 | 9.6286 |
| 65° | 9.6259 | 9.6232 | 9.6205 | 9.6177 | 9.6149 | 9.6121 |
| 66 | 9.6093 | 9.6065 | 9.6036 | 9.6007 | 9.5978 | 9.5948 |
| 67 | 9.5919 | 9.5889 | 9.5859 | 9.5828 | 9.5798 | 9.5767 |
| 68 | 9.5736 | 9.5704 | 9.5673 | 9.5641 | 9.5609 | 9.5576 |
| 69 | 9.5543 | 9.5510 | 9.5477 | 9.5443 | 9.5409 | 9.5375 |
| 70° | 9.5341 | 9.5306 | 9.5270 | 9.5235 | 9.5199 | 9.5163 |
| 71 | 9.5126 | 9.5090 | 9.5052 | 9.5015 | 9.4977 | 9.4939 |
| 72 | 9.4900 | 9.4861 | 9.4821 | 9.4781 | 9.4741 | 9.4700 |
| 73 | 9.4659 | 9.4618 | 9.4576 | 9.4533 | 9.4491 | 9.4447 |
| 74 | 9.4403 | 9.4359 | 9.4314 | 9.4269 | 9.4223 | 9.4177 |
| 75° | 9.4130 | 9.4083 | 9.4035 | 9.3986 | 9.3937 | 9.3887 |
| 76 | 9.3837 | 9.3786 | 9.3734 | 9.3682 | 9.3629 | 9.3575 |
| 77 | 9.3521 | 9.3466 | 9.3410 | 9.3353 | 9.3296 | 9.3238 |
| 78 | 9.3179 | 9.3119 | 9.3058 | 9.2997 | 9.2934 | 9.2870 |
| 79 | 9.2806 | 9.2740 | 9.2674 | 9.2606 | 9.2538 | 9.2468 |
| 80° | 9.2397 | 9.2324 | 9.2251 | 9.2176 | 9.2100 | 9.2022 |
| 81 | 9.1943 | 9.1863 | 9.1781 | 9.1697 | 9.1612 | 9.1525 |
| 82 | 9.1436 | 9.1345 | 9.1252 | 9.1157 | 9.1060 | 9.0961 |
| 83 | 9.0859 | 9.0755 | 9.0648 | 9.0539 | 9.0426 | 9.0311 |
| 84 | 9.0192 | 9.0070 | 8.9945 | 8.9816 | 8.9682 | 8.9545 |
| 85° | 8.9403 | 8.9256 | 8.9104 | 8.8946 | 8.8783 | 8.8613 |
| 86 | 8.8436 | 8.8251 | 8.8059 | 8.7857 | 8.7645 | 8.7423 |
| 87 | 8.7188 | 8.6940 | 8.6677 | 8.6397 | 8.6097 | 8.5776 |
| 88 | 8.5428 | 8.5050 | 8.4637 | 8.4179 | 8.3668 | 8.3088 |
| 89 | 8.2419 | 8.1627 | 8.0658 | 7.9408 | 7.7648 | 7.4637 |
| 90° | | | | | | |

**TABLE**
**12**

# log tan $x$ ($x$ in degrees and minutes)

[subtract 10 from each entry]

| $x$ | 0′ | 10′ | 20′ | 30′ | 40′ | 50′ |
|---|---|---|---|---|---|---|
| 0° | – | 7.4637 | 7.7648 | 7.9409 | 8.0658 | 8.1627 |
| 1 | 8.2419 | 8.3089 | 8.3669 | 8.4181 | 8.4638 | 8.5053 |
| 2 | 8.5431 | 8.5779 | 8.6101 | 8.6401 | 8.6682 | 8.6945 |
| 3 | 8.7194 | 8.7429 | 8.7652 | 8.7865 | 8.8067 | 8.8261 |
| 4 | 8.8446 | 8.8624 | 8.8795 | 8.8960 | 8.9118 | 8.9272 |
| 5° | 8.9420 | 8.9563 | 8.9701 | 8.9836 | 8.9966 | 9.0093 |
| 6 | 9.0216 | 9.0336 | 9.0453 | 9.0567 | 9.0678 | 9.0786 |
| 7 | 9.0891 | 9.0995 | 9.1096 | 9.1194 | 9.1291 | 9.1385 |
| 8 | 9.1478 | 9.1569 | 9.1658 | 9.1745 | 9.1831 | 9.1915 |
| 9 | 9.1997 | 9.2078 | 9.2158 | 9.2236 | 9.2313 | 9.2389 |
| 10° | 9.2463 | 9.2536 | 9.2609 | 9.2680 | 9.2750 | 9.2819 |
| 11 | 9.2887 | 9.2953 | 9.3020 | 9.3085 | 9.3149 | 9.3212 |
| 12 | 9.3275 | 9.3336 | 9.3397 | 9.3458 | 9.3517 | 9.3576 |
| 13 | 9.3634 | 9.3691 | 9.3748 | 9.3804 | 9.3859 | 9.3914 |
| 14 | 9.3968 | 9.4021 | 9.4074 | 9.4127 | 9.4178 | 9.4230 |
| 15° | 9.4281 | 9.4331 | 9.4381 | 9.4430 | 9.4479 | 9.4527 |
| 16 | 9.4575 | 9.4622 | 9.4669 | 9.4716 | 9.4762 | 9.4808 |
| 17 | 9.4853 | 9.4898 | 9.4943 | 9.4987 | 9.5031 | 9.5075 |
| 18 | 9.5118 | 9.5161 | 9.5203 | 9.5245 | 9.5287 | 9.5329 |
| 19 | 9.5370 | 9.5411 | 9.5451 | 9.5491 | 9.5531 | 9.5571 |
| 20° | 9.5611 | 9.5650 | 9.5689 | 9.5727 | 9.5766 | 9.5804 |
| 21 | 9.5842 | 9.5879 | 9.5917 | 9.5954 | 9.5991 | 9.6028 |
| 22 | 9.6064 | 9.6100 | 9.6136 | 9.6172 | 9.6208 | 9.6243 |
| 23 | 9.6279 | 9.6314 | 9.6348 | 9.6383 | 9.6417 | 9.6452 |
| 24 | 9.6486 | 9.6520 | 9.6553 | 9.6587 | 9.6620 | 9.6654 |
| 25° | 9.6687 | 9.6720 | 9.6752 | 9.6785 | 9.6817 | 9.6850 |
| 26 | 9.6882 | 9.6914 | 9.6946 | 9.6977 | 9.7009 | 9.7040 |
| 27 | 9.7072 | 9.7103 | 9.7134 | 9.7165 | 9.7196 | 9.7226 |
| 28 | 9.7257 | 9.7287 | 9.7317 | 9.7348 | 9.7378 | 9.7408 |
| 29 | 9.7438 | 9.7467 | 9.7497 | 9.7526 | 9.7556 | 9.7585 |
| 30° | 9.7614 | 9.7644 | 9.7673 | 9.7701 | 9.7730 | 9.7759 |
| 31 | 9.7788 | 9.7816 | 9.7845 | 9.7873 | 9.7902 | 9.7930 |
| 32 | 9.7958 | 9.7986 | 9.8014 | 9.8042 | 9.8070 | 9.8097 |
| 33 | 9.8125 | 9.8153 | 9.8180 | 9.8208 | 9.8235 | 9.8263 |
| 34 | 9.8290 | 9.8317 | 9.8344 | 9.8371 | 9.8398 | 9.8425 |
| 35° | 9.8452 | 9.8479 | 9.8506 | 9.8533 | 9.8559 | 9.8586 |
| 36 | 9.8613 | 9.8639 | 9.8666 | 9.8692 | 9.8718 | 9.8745 |
| 37 | 9.8771 | 9.8797 | 9.8824 | 9.8850 | 9.8876 | 9.8902 |
| 38 | 9.8928 | 9.8954 | 9.8980 | 9.9006 | 9.9032 | 9.9058 |
| 39 | 9.9084 | 9.9110 | 9.9135 | 9.9161 | 9.9187 | 9.9212 |
| 40° | 9.9238 | 9.9264 | 9.9289 | 9.9315 | 9.9341 | 9.9366 |
| 41 | 9.9392 | 9.9417 | 9.9443 | 9.9468 | 9.9494 | 9.9519 |
| 42 | 9.9544 | 9.9570 | 9.9595 | 9.9621 | 9.9646 | 9.9671 |
| 43 | 9.9697 | 9.9722 | 9.9747 | 9.9772 | 9.9798 | 9.9823 |
| 44 | 9.9848 | 9.9874 | 9.9899 | 9.9924 | 9.9949 | 9.9975 |
| 45° | 10.0000 | 10.0025 | 10.0051 | 10.0076 | 10.0101 | 10.0126 |

**Table 12**
**(continued)**

## log tan $x$ ($x$ in degrees and minutes)

[subtract 10 from each entry]

| $x$ | 0′ | 10′ | 20′ | 30′ | 40′ | 50′ |
|---|---|---|---|---|---|---|
| 45° | 10.0000 | 10.0025 | 10.0051 | 10.0076 | 10.0101 | 10.0126 |
| 46 | 10.0152 | 10.0177 | 10.0202 | 10.0228 | 10.0253 | 10.0278 |
| 47 | 10.0303 | 10.0329 | 10.0354 | 10.0379 | 10.0405 | 10.0430 |
| 48 | 10.0456 | 10.0481 | 10.0506 | 10.0532 | 10.0557 | 10.0583 |
| 49 | 10.0608 | 10.0634 | 10.0659 | 10.0685 | 10.0711 | 10.0736 |
| 50° | 10.0762 | 10.0788 | 10.0813 | 10.0839 | 10.0865 | 10.0890 |
| 51 | 10.0916 | 10.0942 | 10.0968 | 10.0994 | 10.1020 | 10.1046 |
| 52 | 10.1072 | 10.1098 | 10.1124 | 10.1150 | 10.1176 | 10.1203 |
| 53 | 10.1229 | 10.1255 | 10.1282 | 10.1308 | 10.1334 | 10.1361 |
| 54 | 10.1387 | 10.1414 | 10.1441 | 10.1467 | 10.1494 | 10.1521 |
| 55° | 10.1548 | 10.1575 | 10.1602 | 10.1629 | 10.1656 | 10.1683 |
| 56 | 10.1710 | 10.1737 | 10.1765 | 10.1792 | 10.1820 | 10.1847 |
| 57 | 10.1875 | 10.1903 | 10.1930 | 10.1958 | 10.1986 | 10.2014 |
| 58 | 10.2042 | 10.2070 | 10.2098 | 10.2127 | 10.2155 | 10.2184 |
| 59 | 10.2212 | 10.2241 | 10.2270 | 10.2299 | 10.2327 | 10.2356 |
| 60° | 10.2386 | 10.2415 | 10.2444 | 10.2474 | 10.2503 | 10.2533 |
| 61 | 10.2562 | 10.2592 | 10.2622 | 10.2652 | 10.2683 | 10.2713 |
| 62 | 10.2743 | 10.2774 | 10.2804 | 10.2835 | 10.2866 | 10.2897 |
| 63 | 10.2928 | 10.2960 | 10.2991 | 10.3023 | 10.3054 | 10.3086 |
| 64 | 10.3118 | 10.3150 | 10.3183 | 10.3215 | 10.3248 | 10.3280 |
| 65° | 10.3313 | 10.3346 | 10.3380 | 10.3413 | 10.3447 | 10.3480 |
| 66 | 10.3514 | 10.3548 | 10.3583 | 10.3617 | 10.3652 | 10.3686 |
| 67 | 10.3721 | 10.3757 | 10.3792 | 10.3828 | 10.3864 | 10.3900 |
| 68 | 10.3936 | 10.3972 | 10.4009 | 10.4046 | 10.4083 | 10.4121 |
| 69 | 10.4158 | 10.4196 | 10.4234 | 10.4273 | 10.4311 | 10.4350 |
| 70° | 10.4389 | 10.4429 | 10.4469 | 10.4509 | 10.4549 | 10.4589 |
| 71 | 10.4630 | 10.4671 | 10.4713 | 10.4755 | 10.4797 | 10.4839 |
| 72 | 10.4882 | 10.4925 | 10.4969 | 10.5013 | 10.5057 | 10.5102 |
| 73 | 10.5147 | 10.5192 | 10.5238 | 10.5284 | 10.5331 | 10.5378 |
| 74 | 10.5425 | 10.5473 | 10.5521 | 10.5570 | 10.5619 | 10.5669 |
| 75° | 10.5719 | 10.5770 | 10.5822 | 10.5873 | 10.5926 | 10.5979 |
| 76 | 10.6032 | 10.6086 | 10.6141 | 10.6196 | 10.6252 | 10.6309 |
| 77 | 10.6366 | 10.6424 | 10.6483 | 10.6542 | 10.6603 | 10.6664 |
| 78 | 10.6725 | 10.6788 | 10.6851 | 10.6915 | 10.6980 | 10.7047 |
| 79 | 10.7113 | 10.7181 | 10.7250 | 10.7320 | 10.7391 | 10.7464 |
| 80° | 10.7537 | 10.7611 | 10.7687 | 10.7764 | 10.7842 | 10.7922 |
| 81 | 10.8003 | 10.8085 | 10.8169 | 10.8255 | 10.8342 | 10.8431 |
| 82 | 10.8522 | 10.8615 | 10.8709 | 10.8806 | 10.8904 | 10.9005 |
| 83 | 10.9109 | 10.9214 | 10.9322 | 10.9433 | 10.9547 | 10.9664 |
| 84 | 10.9784 | 10.9907 | 11.0034 | 11.0164 | 11.0299 | 11.0437 |
| 85° | 11.0580 | 11.0728 | 11.0882 | 11.1040 | 11.1205 | 11.1376 |
| 86 | 11.1554 | 11.1739 | 11.1933 | 11.2135 | 11.2348 | 11.2571 |
| 87 | 11.2806 | 11.3055 | 11.3318 | 11.3599 | 11.3899 | 11.4221 |
| 88 | 11.4569 | 11.4947 | 11.5362 | 11.5819 | 11.6331 | 11.6911 |
| 89 | 11.7581 | 11.8373 | 11.9342 | 12.0591 | 12.2352 | 12.5363 |
| 90° |  |  |  |  |  |  |

**TABLE**
# 13

# CONVERSION OF RADIANS TO DEGREES, MINUTES AND SECONDS OR FRACTIONS OF DEGREES

| Radians | Deg. | Min. | Sec. | Fractions of Degrees |
|---|---|---|---|---|
| 1 | 57° | 17′ | 44.8″ | 57.2958° |
| 2 | 114° | 35′ | 29.6″ | 114.5916° |
| 3 | 171° | 53′ | 14.4″ | 171.8873° |
| 4 | 229° | 10′ | 59.2″ | 229.1831° |
| 5 | 286° | 28′ | 44.0″ | 286.4789° |
| 6 | 343° | 46′ | 28.8″ | 343.7747° |
| 7 | 401° | 4′ | 13.6″ | 401.0705° |
| 8 | 458° | 21′ | 58.4″ | 458.3662° |
| 9 | 515° | 39′ | 43.3″ | 515.6620° |
| 10 | 572° | 57′ | 28.1″ | 572.9578° |
| .1 | 5° | 43′ | 46.5″ | |
| .2 | 11° | 27′ | 33.0″ | |
| .3 | 17° | 11′ | 19.4″ | |
| .4 | 22° | 55′ | 5.9″ | |
| .5 | 28° | 38′ | 52.4″ | |
| .6 | 34° | 22′ | 38.9″ | |
| .7 | 40° | 6′ | 25.4″ | |
| .8 | 45° | 50′ | 11.8″ | |
| .9 | 51° | 33′ | 58.3″ | |
| .01 | 0° | 34′ | 22.6″ | |
| .02 | 1° | 8′ | 45.3″ | |
| .03 | 1° | 43′ | 7.9″ | |
| .04 | 2° | 17′ | 30.6″ | |
| .05 | 2° | 51′ | 53.2″ | |
| .06 | 3° | 26′ | 15.9″ | |
| .07 | 4° | 0′ | 38.5″ | |
| .08 | 4° | 35′ | 1.2″ | |
| .09 | 5° | 9′ | 23.8″ | |
| .001 | 0° | 3′ | 26.3″ | |
| .002 | 0° | 6′ | 52.5″ | |
| .003 | 0° | 10′ | 18.8″ | |
| .004 | 0° | 13′ | 45.1″ | |
| .005 | 0° | 17′ | 11.3″ | |
| .006 | 0° | 20′ | 37.6″ | |
| .007 | 0° | 24′ | 3.9″ | |
| .008 | 0° | 27′ | 30.1″ | |
| .009 | 0° | 30′ | 56.4″ | |
| .0001 | 0° | 0′ | 20.6″ | |
| .0002 | 0° | 0′ | 41.3″ | |
| .0003 | 0° | 1′ | 1.9″ | |
| .0004 | 0° | 1′ | 22.5″ | |
| .0005 | 0° | 1′ | 43.1″ | |
| .0006 | 0° | 2′ | 3.8″ | |
| .0007 | 0° | 2′ | 24.4″ | |
| .0008 | 0° | 2′ | 45.0″ | |
| .0009 | 0° | 3′ | 5.6″ | |

## TABLE 14

## CONVERSION OF DEGREES, MINUTES AND SECONDS TO RADIANS

| Degrees | Radians |
|---|---|
| 1° | .0174533 |
| 2° | .0349066 |
| 3° | .0523599 |
| 4° | .0698132 |
| 5° | .0872665 |
| 6° | .1047198 |
| 7° | .1221730 |
| 8° | .1396263 |
| 9° | .1570796 |
| 10° | .1745329 |

| Minutes | Radians |
|---|---|
| 1′ | .00029089 |
| 2′ | .00058178 |
| 3′ | .00087266 |
| 4′ | .00116355 |
| 5′ | .00145444 |
| 6′ | .00174533 |
| 7′ | .00203622 |
| 8′ | .00232711 |
| 9′ | .00261800 |
| 10′ | .00290888 |

| Seconds | Radians |
|---|---|
| 1″ | .0000048481 |
| 2″ | .0000096963 |
| 3″ | .0000145444 |
| 4″ | .0000193925 |
| 5″ | .0000242407 |
| 6″ | .0000290888 |
| 7″ | .0000339370 |
| 8″ | .0000387851 |
| 9″ | .0000436332 |
| 10″ | .0000484814 |

TABLE

15

## NATURAL OR NAPIERIAN LOGARITHMS

## $\log_e x$ or $\ln x$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.0 | .00000 | .00995 | .01980 | .02956 | .03922 | .04879 | .05827 | .06766 | .07696 | .08618 |
| 1.1 | .09531 | .10436 | .11333 | .12222 | .13103 | .13976 | .14842 | .15700 | .16551 | .17395 |
| 1.2 | .18232 | .19062 | .19885 | .20701 | .21511 | .22314 | .23111 | .23902 | .24686 | .25464 |
| 1.3 | .26236 | .27003 | .27763 | .28518 | .29267 | .30010 | .30748 | .31481 | .32208 | .32930 |
| 1.4 | .33647 | .34359 | .35066 | .35767 | .36464 | .37156 | .37844 | .38526 | .39204 | .39878 |
| 1.5 | .40547 | .41211 | .41871 | .42527 | .43178 | .43825 | .44469 | .45108 | .45742 | .46373 |
| 1.6 | .47000 | .47623 | .48243 | .48858 | .49470 | .50078 | .50682 | .51282 | .51879 | .52473 |
| 1.7 | .53063 | .53649 | .54232 | .54812 | .55389 | .55962 | .56531 | .57098 | .57661 | .58222 |
| 1.8 | .58779 | .59333 | .59884 | .60432 | .60977 | .61519 | .62058 | .62594 | .63127 | .63658 |
| 1.9 | .64185 | .64710 | .65233 | .65752 | .66269 | .66783 | .67294 | .67803 | .68310 | .68813 |
| 2.0 | .69315 | .69813 | .70310 | .70804 | .71295 | .71784 | .72271 | .72755 | .73237 | .73716 |
| 2.1 | .74194 | .74669 | .75142 | .75612 | .76081 | .76547 | .77011 | .77473 | .77932 | .78390 |
| 2.2 | .78846 | .79299 | .79751 | .80200 | .80648 | .81093 | .81536 | .81978 | .82418 | .82855 |
| 2.3 | .83291 | .83725 | .84157 | .84587 | .85015 | .85442 | .85866 | .86289 | .86710 | .87129 |
| 2.4 | .87547 | .87963 | .88377 | .88789 | .89200 | .89609 | .90016 | .90422 | .90826 | .91228 |
| 2.5 | .91629 | .92028 | .92426 | .92822 | .93216 | .93609 | .94001 | .94391 | .94779 | .95166 |
| 2.6 | .95551 | .95935 | .96317 | .96698 | .97078 | .97456 | .97833 | .98208 | .98582 | .98954 |
| 2.7 | .99325 | .99695 | 1.00063 | 1.00430 | 1.00796 | 1.01160 | 1.01523 | 1.01885 | 1.02245 | 1.02604 |
| 2.8 | 1.02962 | 1.03318 | 1.03674 | 1.04028 | 1.04380 | 1.04732 | 1.05082 | 1.05431 | 1.05779 | 1.06126 |
| 2.9 | 1.06471 | 1.06815 | 1.07158 | 1.07500 | 1.07841 | 1.08181 | 1.08519 | 1.08856 | 1.09192 | 1.09527 |
| 3.0 | 1.09861 | 1.10194 | 1.10526 | 1.10856 | 1.11186 | 1.11514 | 1.11841 | 1.12168 | 1.12493 | 1.12817 |
| 3.1 | 1.13140 | 1.13462 | 1.13783 | 1.14103 | 1.14422 | 1.14740 | 1.15057 | 1.15373 | 1.15688 | 1.16002 |
| 3.2 | 1.16315 | 1.16627 | 1.16938 | 1.17248 | 1.17557 | 1.17865 | 1.18173 | 1.18479 | 1.18784 | 1.19089 |
| 3.3 | 1.19392 | 1.19695 | 1.19996 | 1.20297 | 1.20597 | 1.20896 | 1.21194 | 1.21491 | 1.21788 | 1.22083 |
| 3.4 | 1.22378 | 1.22671 | 1.22964 | 1.23256 | 1.23547 | 1.23837 | 1.24127 | 1.24415 | 1.24703 | 1.24990 |
| 3.5 | 1.25276 | 1.25562 | 1.25846 | 1.26130 | 1.26413 | 1.26695 | 1.26976 | 1.27257 | 1.27536 | 1.27815 |
| 3.6 | 1.28093 | 1.28371 | 1.28647 | 1.28923 | 1.29198 | 1.29473 | 1.29746 | 1.30019 | 1.30291 | 1.30563 |
| 3.7 | 1.30833 | 1.31103 | 1.31372 | 1.31641 | 1.31909 | 1.32176 | 1.32442 | 1.32708 | 1.32972 | 1.33237 |
| 3.8 | 1.33500 | 1.33763 | 1.34025 | 1.34286 | 1.34547 | 1.34807 | 1.35067 | 1.35325 | 1.35584 | 1.35841 |
| 3.9 | 1.36098 | 1.36354 | 1.36609 | 1.36864 | 1.37118 | 1.37372 | 1.37624 | 1.37877 | 1.38128 | 1.38379 |
| 4.0 | 1.38629 | 1.38879 | 1.39128 | 1.39377 | 1.39624 | 1.39872 | 1.40118 | 1.40364 | 1.40610 | 1.40854 |
| 4.1 | 1.41099 | 1.41342 | 1.41585 | 1.41828 | 1.42070 | 1.42311 | 1.42552 | 1.42792 | 1.43031 | 1.43270 |
| 4.2 | 1.43508 | 1.43746 | 1.43984 | 1.44220 | 1.44456 | 1.44692 | 1.44927 | 1.45161 | 1.45395 | 1.45629 |
| 4.3 | 1.45862 | 1.46094 | 1.46326 | 1.46557 | 1.46787 | 1.47018 | 1.47247 | 1.47476 | 1.47705 | 1.47933 |
| 4.4 | 1.48160 | 1.48387 | 1.48614 | 1.48840 | 1.49065 | 1.49290 | 1.49515 | 1.49739 | 1.49962 | 1.50185 |
| 4.5 | 1.50408 | 1.50630 | 1.50851 | 1.51072 | 1.51293 | 1.51513 | 1.51732 | 1.51951 | 1.52170 | 1.52388 |
| 4.6 | 1.52606 | 1.52823 | 1.53039 | 1.53256 | 1.53471 | 1.53687 | 1.53902 | 1.54116 | 1.54330 | 1.54543 |
| 4.7 | 1.54756 | 1.54969 | 1.55181 | 1.55393 | 1.55604 | 1.55814 | 1.56025 | 1.56235 | 1.56444 | 1.56653 |
| 4.8 | 1.56862 | 1.57070 | 1.57277 | 1.57485 | 1.57691 | 1.57898 | 1.58104 | 1.58309 | 1.58515 | 1.58719 |
| 4.9 | 1.58924 | 1.59127 | 1.59331 | 1.59534 | 1.59737 | 1.59939 | 1.60141 | 1.60342 | 1.60543 | 1.60744 |

ln 10 = 2.30259

2 ln 10 = 4.60517

3 ln 10 = 6.90776

4 ln 10 = 9.21034

5 ln 10 = 11.51293

6 ln 10 = 13.81551

7 ln 10 = 16.11810

8 ln 10 = 18.42068

9 ln 10 = 20.72327

**Table 15**
(continued)

## NATURAL OR NAPIERIAN LOGARITHMS

## $\log_e x$ or $\ln x$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 5.0 | 1.60944 | 1.61144 | 1.61343 | 1.61542 | 1.61741 | 1.61939 | 1.62137 | 1.62334 | 1.62531 | 1.62728 |
| 5.1 | 1.62924 | 1.63120 | 1.63315 | 1.63511 | 1.63705 | 1.63900 | 1.64094 | 1.64287 | 1.64481 | 1.64673 |
| 5.2 | 1.64866 | 1.65058 | 1.65250 | 1.65441 | 1.65632 | 1.65823 | 1.66013 | 1.66203 | 1.66393 | 1.66582 |
| 5.3 | 1.66771 | 1.66959 | 1.67147 | 1.67335 | 1.67523 | 1.67710 | 1.67896 | 1.68083 | 1.68269 | 1.68455 |
| 5.4 | 1.68640 | 1.68825 | 1.69010 | 1.69194 | 1.69378 | 1.69562 | 1.69745 | 1.69928 | 1.70111 | 1.70293 |
| 5.5 | 1.70475 | 1.70656 | 1.70838 | 1.71019 | 1.71199 | 1.71380 | 1.71560 | 1.71740 | 1.71919 | 1.72098 |
| 5.6 | 1.72277 | 1.72455 | 1.72633 | 1.72811 | 1.72988 | 1.73166 | 1.73342 | 1.73519 | 1.73695 | 1.73871 |
| 5.7 | 1.74047 | 1.74222 | 1.74397 | 1.74572 | 1.74746 | 1.74920 | 1.75094 | 1.75267 | 1.75440 | 1.75613 |
| 5.8 | 1.75786 | 1.75958 | 1.76130 | 1.76302 | 1.76473 | 1.76644 | 1.76815 | 1.76985 | 1.77156 | 1.77326 |
| 5.9 | 1.77495 | 1.77665 | 1.77834 | 1.78002 | 1.78171 | 1.78339 | 1.78507 | 1.78675 | 1.78842 | 1.79009 |
| 6.0 | 1.79176 | 1.79342 | 1.79509 | 1.79675 | 1.79840 | 1.80006 | 1.80171 | 1.80336 | 1.80500 | 1.80665 |
| 6.1 | 1.80829 | 1.80993 | 1.81156 | 1.81319 | 1.81482 | 1.81645 | 1.81808 | 1.81970 | 1.82132 | 1.82294 |
| 6.2 | 1.82455 | 1.82616 | 1.82777 | 1.82938 | 1.83098 | 1.83258 | 1.83418 | 1.83578 | 1.83737 | 1.83896 |
| 6.3 | 1.84055 | 1.84214 | 1.84372 | 1.84530 | 1.84688 | 1.84845 | 1.85003 | 1.85160 | 1.85317 | 1.85473 |
| 6.4 | 1.85630 | 1.85786 | 1.85942 | 1.86097 | 1.86253 | 1.86408 | 1.86563 | 1.86718 | 1.86872 | 1.87026 |
| 6.5 | 1.87180 | 1.87334 | 1.87487 | 1.87641 | 1.87794 | 1.87947 | 1.88099 | 1.88251 | 1.88403 | 1.88555 |
| 6.6 | 1.88707 | 1.88858 | 1.89010 | 1.89160 | 1.89311 | 1.89462 | 1.89612 | 1.89762 | 1.89912 | 1.90061 |
| 6.7 | 1.90211 | 1.90360 | 1.90509 | 1.90658 | 1.90806 | 1.90954 | 1.91102 | 1.91250 | 1.91398 | 1.91545 |
| 6.8 | 1.91692 | 1.91839 | 1.91986 | 1.92132 | 1.92279 | 1.92425 | 1.92571 | 1.92716 | 1.92862 | 1.93007 |
| 6.9 | 1.93152 | 1.93297 | 1.93442 | 1.93586 | 1.93730 | 1.93874 | 1.94018 | 1.94162 | 1.94305 | 1.94448 |
| 7.0 | 1.94591 | 1.94734 | 1.94876 | 1.95019 | 1.95161 | 1.95303 | 1.95445 | 1.95586 | 1.95727 | 1.95869 |
| 7.1 | 1.96009 | 1.96150 | 1.96291 | 1.96431 | 1.96571 | 1.96711 | 1.96851 | 1.96991 | 1.97130 | 1.97269 |
| 7.2 | 1.97408 | 1.97547 | 1.97685 | 1.97824 | 1.97962 | 1.98100 | 1.98238 | 1.98376 | 1.98513 | 1.98650 |
| 7.3 | 1.98787 | 1.98924 | 1.99061 | 1.99198 | 1.99334 | 1.99470 | 1.99606 | 1.99742 | 1.99877 | 2.00013 |
| 7.4 | 2.00148 | 2.00283 | 2.00418 | 2.00553 | 2.00687 | 2.00821 | 2.00956 | 2.01089 | 2.01223 | 2.01357 |
| 7.5 | 2.01490 | 2.01624 | 2.01757 | 2.01890 | 2.02022 | 2.02155 | 2.02287 | 2.02419 | 2.02551 | 2.02683 |
| 7.6 | 2.02815 | 2.02946 | 2.03078 | 2.03209 | 2.03340 | 2.03471 | 2.03601 | 2.03732 | 2.03862 | 2.03992 |
| 7.7 | 2.04122 | 2.04252 | 2.04381 | 2.04511 | 2.04640 | 2.04769 | 2.04898 | 2.05027 | 2.05156 | 2.05284 |
| 7.8 | 2.05412 | 2.05540 | 2.05668 | 2.05796 | 2.05924 | 2.06051 | 2.06179 | 2.06306 | 2.06433 | 2.06560 |
| 7.9 | 2.06686 | 2.06813 | 2.06939 | 2.07065 | 2.07191 | 2.07317 | 2.07443 | 2.07568 | 2.07694 | 2.07819 |
| 8.0 | 2.07944 | 2.08069 | 2.08194 | 2.08318 | 2.08443 | 2.08567 | 2.08691 | 2.08815 | 2.08939 | 2.09063 |
| 8.1 | 2.09186 | 2.09310 | 2.09433 | 2.09556 | 2.09679 | 2.09802 | 2.09924 | 2.10047 | 2.10169 | 2.10291 |
| 8.2 | 2.10413 | 2.10535 | 2.10657 | 2.10779 | 2.10900 | 2.11021 | 2.11142 | 2.11263 | 2.11384 | 2.11505 |
| 8.3 | 2.11626 | 2.11746 | 2.11866 | 2.11986 | 2.12106 | 2.12226 | 2.12346 | 2.12465 | 2.12585 | 2.12704 |
| 8.4 | 2.12823 | 2.12942 | 2.13061 | 2.13180 | 2.13298 | 2.13417 | 2.13535 | 2.13653 | 2.13771 | 2.13889 |
| 8.5 | 2.14007 | 2.14124 | 2.14242 | 2.14359 | 2.14476 | 2.14593 | 2.14710 | 2.14827 | 2.14943 | 2.15060 |
| 8.6 | 2.15176 | 2.15292 | 2.15409 | 2.15524 | 2.15640 | 2.15756 | 2.15871 | 2.15987 | 2.16102 | 2.16217 |
| 8.7 | 2.16332 | 2.16447 | 2.16562 | 2.16677 | 2.16791 | 2.16905 | 2.17020 | 2.17134 | 2.17248 | 2.17361 |
| 8.8 | 2.17475 | 2.17589 | 2.17702 | 2.17816 | 2.17929 | 2.18042 | 2.18155 | 2.18267 | 2.18380 | 2.18493 |
| 8.9 | 2.18605 | 2.18717 | 2.18830 | 2.18942 | 2.19054 | 2.19165 | 2.19277 | 2.19389 | 2.19500 | 2.19611 |
| 9.0 | 2.19722 | 2.19834 | 2.19944 | 2.20055 | 2.20166 | 2.20276 | 2.20387 | 2.20497 | 2.20607 | 2.20717 |
| 9.1 | 2.20827 | 2.20937 | 2.21047 | 2.21157 | 2.21266 | 2.21375 | 2.21485 | 2.21594 | 2.21703 | 2.21812 |
| 9.2 | 2.21920 | 2.22029 | 2.22138 | 2.22246 | 2.22354 | 2.22462 | 2.22570 | 2.22678 | 2.22786 | 2.22894 |
| 9.3 | 2.23001 | 2.23109 | 2.23216 | 2.23324 | 2.23431 | 2.23538 | 2.23645 | 2.23751 | 2.23858 | 2.23965 |
| 9.4 | 2.24071 | 2.24177 | 2.24284 | 2.24390 | 2.24496 | 2.24601 | 2.24707 | 2.24813 | 2.24918 | 2.25024 |
| 9.5 | 2.25129 | 2.25234 | 2.25339 | 2.25444 | 2.25549 | 2.25654 | 2.25759 | 2.25863 | 2.25968 | 2.26072 |
| 9.6 | 2.26176 | 2.26280 | 2.26384 | 2.26488 | 2.26592 | 2.26696 | 2.26799 | 2.26903 | 2.27006 | 2.27109 |
| 9.7 | 2.27213 | 2.27316 | 2.27419 | 2.27521 | 2.27624 | 2.27727 | 2.27829 | 2.27932 | 2.28034 | 2.28136 |
| 9.8 | 2.28238 | 2.28340 | 2.28442 | 2.28544 | 2.28646 | 2.28747 | 2.28849 | 2.28950 | 2.29051 | 2.29152 |
| 9.9 | 2.29253 | 2.29354 | 2.29455 | 2.29556 | 2.29657 | 2.29757 | 2.29858 | 2.29958 | 2.30058 | 2.30158 |

**TABLE**
**16**

## EXPONENTIAL FUNCTIONS
## $e^x$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| .0 | 1.0000 | 1.0101 | 1.0202 | 1.0305 | 1.0408 | 1.0513 | 1.0618 | 1.0725 | 1.0833 | 1.0942 |
| .1 | 1.1052 | 1.1163 | 1.1275 | 1.1388 | 1.1503 | 1.1618 | 1.1735 | 1.1853 | 1.1972 | 1.2092 |
| .2 | 1.2214 | 1.2337 | 1.2461 | 1.2586 | 1.2712 | 1.2840 | 1.2969 | 1.3100 | 1.3231 | 1.3364 |
| .3 | 1.3499 | 1.3634 | 1.3771 | 1.3910 | 1.4049 | 1.4191 | 1.4333 | 1.4477 | 1.4623 | 1.4770 |
| .4 | 1.4918 | 1.5068 | 1.5220 | 1.5373 | 1.5527 | 1.5683 | 1.5841 | 1.6000 | 1.6161 | 1.6323 |
| .5 | 1.6487 | 1.6653 | 1.6820 | 1.6989 | 1.7160 | 1.7333 | 1.7507 | 1.7683 | 1.7860 | 1.8040 |
| .6 | 1.8221 | 1.8404 | 1.8589 | 1.8776 | 1.8965 | 1.9155 | 1.9348 | 1.9542 | 1.9739 | 1.9937 |
| .7 | 2.0138 | 2.0340 | 2.0544 | 2.0751 | 2.0959 | 2.1170 | 2.1383 | 2.1598 | 2.1815 | 2.2034 |
| .8 | 2.2255 | 2.2479 | 2.2705 | 2.2933 | 2.3164 | 2.3396 | 2.3632 | 2.3869 | 2.4109 | 2.4351 |
| .9 | 2.4596 | 2.4843 | 2.5093 | 2.5345 | 2.5600 | 2.5857 | 2.6117 | 2.6379 | 2.6645 | 2.6912 |
| 1.0 | 2.7183 | 2.7456 | 2.7732 | 2.8011 | 2.8292 | 2.8577 | 2.8864 | 2.9154 | 2.9447 | 2.9743 |
| 1.1 | 3.0042 | 3.0344 | 3.0649 | 3.0957 | 3.1268 | 3.1582 | 3.1899 | 3.2220 | 3.2544 | 3.2871 |
| 1.2 | 3.3201 | 3.3535 | 3.3872 | 3.4212 | 3.4556 | 3.4903 | 3.5254 | 3.5609 | 3.5966 | 3.6328 |
| 1.3 | 3.6693 | 3.7062 | 3.7434 | 3.7810 | 3.8190 | 3.8574 | 3.8962 | 3.9354 | 3.9749 | 4.0149 |
| 1.4 | 4.0552 | 4.0960 | 4.1371 | 4.1787 | 4.2207 | 4.2631 | 4.3060 | 4.3492 | 4.3929 | 4.4371 |
| 1.5 | 4.4817 | 4.5267 | 4.5722 | 4.6182 | 4.6646 | 4.7115 | 4.7588 | 4.8066 | 4.8550 | 4.9037 |
| 1.6 | 4.9530 | 5.0028 | 5.0531 | 5.1039 | 5.1552 | 5.2070 | 5.2593 | 5.3122 | 5.3656 | 5.4195 |
| 1.7 | 5.4739 | 5.5290 | 5.5845 | 5.6407 | 5.6973 | 5.7546 | 5.8124 | 5.8709 | 5.9299 | 5.9895 |
| 1.8 | 6.0496 | 6.1104 | 6.1719 | 6.2339 | 6.2965 | 6.3598 | 6.4237 | 6.4883 | 6.5535 | 6.6194 |
| 1.9 | 6.6859 | 6.7531 | 6.8210 | 6.8895 | 6.9588 | 7.0287 | 7.0993 | 7.1707 | 7.2427 | 7.3155 |
| 2.0 | 7.3891 | 7.4633 | 7.5383 | 7.6141 | 7.6906 | 7.7679 | 7.8460 | 7.9248 | 8.0045 | 8.0849 |
| 2.1 | 8.1662 | 8.2482 | 8.3311 | 8.4149 | 8.4994 | 8.5849 | 8.6711 | 8.7583 | 8.8463 | 8.9352 |
| 2.2 | 9.0250 | 9.1157 | 9.2073 | 9.2999 | 9.3933 | 9.4877 | 9.5831 | 9.6794 | 9.7767 | 9.8749 |
| 2.3 | 9.9742 | 10.074 | 10.176 | 10.278 | 10.381 | 10.486 | 10.591 | 10.697 | 10.805 | 10.913 |
| 2.4 | 11.023 | 11.134 | 11.246 | 11.359 | 11.473 | 11.588 | 11.705 | 11.822 | 11.941 | 12.061 |
| 2.5 | 12.182 | 12.305 | 12.429 | 12.554 | 12.680 | 12.807 | 12.936 | 13.066 | 13.197 | 13.330 |
| 2.6 | 13.464 | 13.599 | 13.736 | 13.874 | 14.013 | 14.154 | 14.296 | 14.440 | 14.585 | 14.732 |
| 2.7 | 14.880 | 15.029 | 15.180 | 15.333 | 15.487 | 15.643 | 15.800 | 15.959 | 16.119 | 16.281 |
| 2.8 | 16.445 | 16.610 | 16.777 | 16.945 | 17.116 | 17.288 | 17.462 | 17.637 | 17.814 | 17.993 |
| 2.9 | 18.174 | 18.357 | 18.541 | 18.728 | 18.916 | 19.106 | 19.298 | 19.492 | 19.688 | 19.886 |
| 3.0 | 20.086 | 20.287 | 20.491 | 20.697 | 20.905 | 21.115 | 21.328 | 21.542 | 21.758 | 21.977 |
| 3.1 | 22.198 | 22.421 | 22.646 | 22.874 | 23.104 | 23.336 | 23.571 | 23.807 | 24.047 | 24.288 |
| 3.2 | 24.533 | 24.779 | 25.028 | 25.280 | 25.534 | 25.790 | 26.050 | 26.311 | 26.576 | 26.843 |
| 3.3 | 27.113 | 27.385 | 27.660 | 27.938 | 28.219 | 28.503 | 28.789 | 29.079 | 29.371 | 29.666 |
| 3.4 | 29.964 | 30.265 | 30.569 | 30.877 | 31.187 | 31.500 | 31.817 | 32.137 | 32.460 | 32.786 |
| 3.5 | 33.115 | 33.448 | 33.784 | 34.124 | 34.467 | 34.813 | 35.163 | 35.517 | 35.874 | 36.234 |
| 3.6 | 36.598 | 36.966 | 37.338 | 37.713 | 38.092 | 38.475 | 38.861 | 39.252 | 39.646 | 40.045 |
| 3.7 | 40.447 | 40.854 | 41.264 | 41.679 | 42.098 | 42.521 | 42.948 | 43.380 | 43.816 | 44.256 |
| 3.8 | 44.701 | 45.150 | 45.604 | 46.063 | 46.525 | 46.993 | 47.465 | 47.942 | 48.424 | 48.911 |
| 3.9 | 49.402 | 49.899 | 50.400 | 50.907 | 51.419 | 51.935 | 52.457 | 52.985 | 53.517 | 54.055 |
| 4. | 54.598 | 60.340 | 66.686 | 73.700 | 81.451 | 90.017 | 99.484 | 109.95 | 121.51 | 134.29 |
| 5. | 148.41 | 164.02 | 181.27 | 200.34 | 221.41 | 244.69 | 270.43 | 298.87 | 330.30 | 365.04 |
| 6. | 403.43 | 445.86 | 492.75 | 544.57 | 601.85 | 665.14 | 735.10 | 812.41 | 897.85 | 992.27 |
| 7. | 1096.6 | 1212.0 | 1339.4 | 1480.3 | 1636.0 | 1808.0 | 1998.2 | 2208.3 | 2440.6 | 2697.3 |
| 8. | 2981.0 | 3294.5 | 3641.0 | 4023.9 | 4447.1 | 4914.8 | 5431.7 | 6002.9 | 6634.2 | 7332.0 |
| 9. | 8103.1 | 8955.3 | 9897.1 | 10938 | 12088 | 13360 | 14765 | 16318 | 18034 | 19930 |
| 10. | 22026 |  |  |  |  |  |  |  |  |  |

## TABLE 17

## EXPONENTIAL FUNCTIONS

$e^{-x}$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| .0 | 1.00000 | .99005 | .98020 | .97045 | .96079 | .95123 | .94176 | .93239 | .92312 | .91393 |
| .1 | .90484 | .89583 | .88692 | .87810 | .86936 | .86071 | .85214 | .84366 | .83527 | .82696 |
| .2 | .81873 | .81058 | .80252 | .79453 | .78663 | .77880 | .77105 | .76338 | .75578 | .74826 |
| .3 | .74082 | .73345 | .72615 | .71892 | .71177 | .70469 | .69768 | .69073 | .68386 | .67706 |
| .4 | .67032 | .66365 | .65705 | .65051 | .64404 | .63763 | .63128 | .62500 | .61878 | .61263 |
| .5 | .60653 | .60050 | .59452 | .58860 | .58275 | .57695 | .57121 | .56553 | .55990 | .55433 |
| .6 | .54881 | .54335 | .53794 | .53259 | .52729 | .52205 | .51685 | .51171 | .50662 | .50158 |
| .7 | .49659 | .49164 | .48675 | .48191 | .47711 | .47237 | .46767 | .46301 | .45841 | .45384 |
| .8 | .44933 | .44486 | .44043 | .43605 | .43171 | .42741 | .42316 | .41895 | .41478 | .41066 |
| .9 | .40657 | .40252 | .39852 | .39455 | .39063 | .38674 | .38289 | .37908 | .37531 | .37158 |
| 1.0 | .36788 | .36422 | .36060 | .35701 | .35345 | .34994 | .34646 | .34301 | .33960 | .33622 |
| 1.1 | .33287 | .32956 | .32628 | .32303 | .31982 | .31664 | .31349 | .31037 | .30728 | .30422 |
| 1.2 | .30119 | .29820 | .29523 | .29229 | .28938 | .28650 | .28365 | .28083 | .27804 | .27527 |
| 1.3 | .27253 | .26982 | .26714 | .26448 | .26185 | .25924 | .25666 | .25411 | .25158 | .24908 |
| 1.4 | .24660 | .24414 | .24171 | .23931 | .23693 | .23457 | .23224 | .22993 | .22764 | .22537 |
| 1.5 | .22313 | .22091 | .21871 | .21654 | .21438 | .21225 | .21014 | .20805 | .20598 | .20393 |
| 1.6 | .20190 | .19989 | .19790 | .19593 | .19398 | .19205 | .19014 | .18825 | .18637 | .18452 |
| 1.7 | .18268 | .18087 | .17907 | .17728 | .17552 | .17377 | .17204 | .17033 | .16864 | .16696 |
| 1.8 | .16530 | .16365 | .16203 | .16041 | .15882 | .15724 | .15567 | .15412 | .15259 | .15107 |
| 1.9 | .14957 | .14808 | .14661 | .14515 | .14370 | .14227 | .14086 | .13946 | .13807 | .13670 |
| 2.0 | .13534 | .13399 | .13266 | .13134 | .13003 | .12873 | .12745 | .12619 | .12493 | .12369 |
| 2.1 | .12246 | .12124 | .12003 | .11884 | .11765 | .11648 | .11533 | .11418 | .11304 | .11192 |
| 2.2 | .11080 | .10970 | .10861 | .10753 | .10646 | .10540 | .10435 | .10331 | .10228 | .10127 |
| 2.3 | .10026 | .09926 | .09827 | .09730 | .09633 | .09537 | .09442 | .09348 | .09255 | .09163 |
| 2.4 | .09072 | .08982 | .08892 | .08804 | .08716 | .08629 | .08543 | .08458 | .08374 | .08291 |
| 2.5 | .08208 | .08127 | .08046 | .07966 | .07887 | .07808 | .07730 | .07654 | .07577 | .07502 |
| 2.6 | .07427 | .07353 | .07280 | .07208 | .07136 | .07065 | .06995 | .06925 | .06856 | .06788 |
| 2.7 | .06721 | .06654 | .06587 | .06522 | .06457 | .06393 | .06329 | .06266 | .06204 | .06142 |
| 2.8 | .06081 | .06020 | .05961 | .05901 | .05843 | .05784 | .05727 | .05670 | .05613 | .05558 |
| 2.9 | .05502 | .05448 | .05393 | .05340 | .05287 | .05234 | .05182 | .05130 | .05079 | .05029 |
| 3.0 | .04979 | .04929 | .04880 | .04832 | .04783 | .04736 | .04689 | .04642 | .04596 | .04550 |
| 3.1 | .04505 | .04460 | .04416 | .04372 | .04328 | .04285 | .04243 | .04200 | .04159 | .04117 |
| 3.2 | .04076 | .04036 | .03996 | .03956 | .03916 | .03877 | .03839 | .03801 | .03763 | .03725 |
| 3.3 | .03688 | .03652 | .03615 | .03579 | .03544 | .03508 | .03474 | .03439 | .03405 | .03371 |
| 3.4 | .03337 | .03304 | .03271 | .03239 | .03206 | .03175 | .03143 | .03112 | .03081 | .03050 |
| 3.5 | .03020 | .02990 | .02960 | .02930 | .02901 | .02872 | .02844 | .02816 | .02788 | .02760 |
| 3.6 | .02732 | .02705 | .02678 | .02652 | .02625 | .02599 | .02573 | .02548 | .02522 | .02497 |
| 3.7 | .02472 | .02448 | .02423 | .02399 | .02375 | .02352 | .02328 | .02305 | .02282 | .02260 |
| 3.8 | .02237 | .02215 | .02193 | .02171 | .02149 | .02128 | .02107 | .02086 | .02065 | .02045 |
| 3.9 | .02024 | .02004 | .01984 | .01964 | .01945 | .01925 | .01906 | .01887 | .01869 | .01850 |
| 4. | .018316 | .016573 | .014996 | .013569 | .012277 | .011109 | .010052 | $.0^{2}90953$ | $.0^{2}82297$ | $.0^{2}74466$ |
| 5. | $.0^{2}67379$ | $.0^{2}60967$ | $.0^{2}55166$ | $.0^{2}49916$ | $.0^{2}45166$ | $.0^{2}40868$ | $.0^{2}36979$ | $.0^{2}33460$ | $.0^{2}30276$ | $.0^{2}27394$ |
| 6. | $.0^{2}24788$ | $.0^{2}22429$ | $.0^{2}20294$ | $.0^{2}18363$ | $.0^{2}16616$ | $.0^{2}15034$ | $.0^{2}13604$ | $.0^{2}12309$ | $.0^{2}11138$ | $.0^{2}10078$ |
| 7. | $.0^{3}91188$ | $.0^{3}82510$ | $.0^{3}74659$ | $.0^{3}67554$ | $.0^{3}61125$ | $.0^{3}55308$ | $.0^{3}50045$ | $.0^{3}45283$ | $.0^{3}40973$ | $.0^{3}37074$ |
| 8. | $.0^{3}33546$ | $.0^{3}30354$ | $.0^{3}27465$ | $.0^{3}24852$ | $.0^{3}22487$ | $.0^{3}20347$ | $.0^{3}18411$ | $.0^{3}16659$ | $.0^{3}15073$ | $.0^{3}13639$ |
| 9. | $.0^{3}12341$ | $.0^{3}11167$ | $.0^{3}10104$ | $.0^{4}91424$ | $.0^{4}82724$ | $.0^{4}74852$ | $.0^{4}67729$ | $.0^{4}61283$ | $.0^{4}55452$ | $.0^{4}50175$ |
| 10. | $.0^{4}45400$ | | | | | | | | | |

## TABLE 18a

## HYPERBOLIC FUNCTIONS

## sinh $x$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| .0 | .0000 | .0100 | .0200 | .0300 | .0400 | .0500 | .0600 | .0701 | .0801 | .0901 |
| .1 | .1002 | .1102 | .1203 | .1304 | .1405 | .1506 | .1607 | .1708 | .1810 | .1911 |
| .2 | .2013 | .2115 | .2218 | .2320 | .2423 | .2526 | .2629 | .2733 | .2837 | .2941 |
| .3 | .3045 | .3150 | .3255 | .3360 | .3466 | .3572 | .3678 | .3785 | .3892 | .4000 |
| .4 | .4108 | .4216 | .4325 | .4434 | .4543 | .4653 | .4764 | .4875 | .4986 | .5098 |
| .5 | .5211 | .5324 | .5438 | .5552 | .5666 | .5782 | .5897 | .6014 | .6131 | .6248 |
| .6 | .6367 | .6485 | .6605 | .6725 | .6846 | .6967 | .7090 | .7213 | .7336 | .7461 |
| .7 | .7586 | .7712 | .7838 | .7966 | .8094 | .8223 | .8353 | .8484 | .8615 | .8748 |
| .8 | .8881 | .9015 | .9150 | .9286 | .9423 | .9561 | .9700 | .9840 | .9981 | 1.0122 |
| .9 | 1.0265 | 1.0409 | 1.0554 | 1.0700 | 1.0847 | 1.0995 | 1.1144 | 1.1294 | 1.1446 | 1.1598 |
| 1.0 | 1.1752 | 1.1907 | 1.2063 | 1.2220 | 1.2379 | 1.2539 | 1.2700 | 1.2862 | 1.3025 | 1.3190 |
| 1.1 | 1.3356 | 1.3524 | 1.3693 | 1.3863 | 1.4035 | 1.4208 | 1.4382 | 1.4558 | 1.4735 | 1.4914 |
| 1.2 | 1.5095 | 1.5276 | 1.5460 | 1.5645 | 1.5831 | 1.6019 | 1.6209 | 1.6400 | 1.6593 | 1.6788 |
| 1.3 | 1.6984 | 1.7182 | 1.7381 | 1.7583 | 1.7786 | 1.7991 | 1.8198 | 1.8406 | 1.8617 | 1.8829 |
| 1.4 | 1.9043 | 1.9259 | 1.9477 | 1.9697 | 1.9919 | 2.0143 | 2.0369 | 2.0597 | 2.0827 | 2.1059 |
| 1.5 | 2.1293 | 2.1529 | 2.1768 | 2.2008 | 2.2251 | 2.2496 | 2.2743 | 2.2993 | 2.3245 | 2.3499 |
| 1.6 | 2.3756 | 2.4015 | 2.4276 | 2.4540 | 2.4806 | 2.5075 | 2.5346 | 2.5620 | 2.5896 | 2.6175 |
| 1.7 | 2.6456 | 2.6740 | 2.7027 | 2.7317 | 2.7609 | 2.7904 | 2.8202 | 2.8503 | 2.8806 | 2.9112 |
| 1.8 | 2.9422 | 2.9734 | 3.0049 | 3.0367 | 3.0689 | 3.1013 | 3.1340 | 3.1671 | 3.2005 | 3.2341 |
| 1.9 | 3.2682 | 3.3025 | 3.3372 | 3.3722 | 3.4075 | 3.4432 | 3.4792 | 3.5156 | 3.5523 | 3.5894 |
| 2.0 | 3.6269 | 3.6647 | 3.7028 | 3.7414 | 3.7803 | 3.8196 | 3.8593 | 3.8993 | 3.9398 | 3.9806 |
| 2.1 | 4.0219 | 4.0635 | 4.1056 | 4.1480 | 4.1909 | 4.2342 | 4.2779 | 4.3221 | 4.3666 | 4.4116 |
| 2.2 | 4.4571 | 4.5030 | 4.5494 | 4.5962 | 4.6434 | 4.6912 | 4.7394 | 4.7880 | 4.8372 | 4.8868 |
| 2.3 | 4.9370 | 4.9876 | 5.0387 | 5.0903 | 5.1425 | 5.1951 | 5.2483 | 5.3020 | 5.3562 | 5.4109 |
| 2.4 | 5.4662 | 5.5221 | 5.5785 | 5.6354 | 5.6929 | 5.7510 | 5.8097 | 5.8689 | 5.9288 | 5.9892 |
| 2.5 | 6.0502 | 6.1118 | 6.1741 | 6.2369 | 6.3004 | 6.3645 | 6.4293 | 6.4946 | 6.5607 | 6.6274 |
| 2.6 | 6.6947 | 6.7628 | 6.8315 | 6.9008 | 6.9709 | 7.0417 | 7.1132 | 7.1854 | 7.2583 | 7.3319 |
| 2.7 | 7.4063 | 7.4814 | 7.5572 | 7.6338 | 7.7112 | 7.7894 | 7.8683 | 7.9480 | 8.0285 | 8.1098 |
| 2.8 | 8.1919 | 8.2749 | 8.3586 | 8.4432 | 8.5287 | 8.6150 | 8.7021 | 8.7902 | 8.8791 | 8.9689 |
| 2.9 | 9.0596 | 9.1512 | 9.2437 | 9.3371 | 9.4315 | 9.5268 | 9.6231 | 9.7203 | 9.8185 | 9.9177 |

**Table 18a**
(continued)

## HYPERBOLIC FUNCTIONS

## sinh $x$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 3.0 | 10.018 | 10.119 | 10.221 | 10.324 | 10.429 | 10.534 | 10.640 | 10.748 | 10.856 | 10.966 |
| 3.1 | 11.076 | 11.188 | 11.301 | 11.415 | 11.530 | 11.647 | 11.764 | 11.883 | 12.003 | 12.124 |
| 3.2 | 12.246 | 12.369 | 12.494 | 12.620 | 12.747 | 12.876 | 13.006 | 13.137 | 13.269 | 13.403 |
| 3.3 | 13.538 | 13.674 | 13.812 | 13.951 | 14.092 | 14.234 | 14.377 | 14.522 | 14.668 | 14.816 |
| 3.4 | 14.965 | 15.116 | 15.268 | 15.422 | 15.577 | 15.734 | 15.893 | 16.053 | 16.215 | 16.378 |
| 3.5 | 16.543 | 16.709 | 16.877 | 17.047 | 17.219 | 17.392 | 17.567 | 17.744 | 17.923 | 18.103 |
| 3.6 | 18.285 | 18.470 | 18.655 | 18.843 | 19.033 | 19.224 | 19.418 | 19.613 | 19.811 | 20.010 |
| 3.7 | 20.211 | 20.415 | 20.620 | 20.828 | 21.037 | 21.249 | 21.463 | 21.679 | 21.897 | 22.117 |
| 3.8 | 22.339 | 22.564 | 22.791 | 23.020 | 23.252 | 23.486 | 23.722 | 23.961 | 24.202 | 24.445 |
| 3.9 | 24.691 | 24.939 | 25.190 | 25.444 | 25.700 | 25.958 | 26.219 | 26.483 | 26.749 | 27.018 |
| 4.0 | 27.290 | 27.564 | 27.842 | 28.122 | 28.404 | 28.690 | 28.979 | 29.270 | 29.564 | 29.862 |
| 4.1 | 30.162 | 30.465 | 30.772 | 31.081 | 31.393 | 31.709 | 32.028 | 32.350 | 32.675 | 33.004 |
| 4.2 | 33.336 | 33.671 | 34.009 | 34.351 | 34.697 | 35.046 | 35.398 | 35.754 | 36.113 | 36.476 |
| 4.3 | 36.843 | 37.214 | 37.588 | 37.965 | 38.347 | 38.733 | 39.122 | 39.515 | 39.913 | 40.314 |
| 4.4 | 40.719 | 41.129 | 41.542 | 41.960 | 42.382 | 42.808 | 43.238 | 43.673 | 44.112 | 44.555 |
| 4.5 | 45.003 | 45.455 | 45.912 | 46.374 | 46.840 | 47.311 | 47.787 | 48.267 | 48.752 | 49.242 |
| 4.6 | 49.737 | 50.237 | 50.742 | 51.252 | 51.767 | 52.288 | 52.813 | 53.344 | 53.880 | 54.422 |
| 4.7 | 54.969 | 55.522 | 56.080 | 56.643 | 57.213 | 57.788 | 58.369 | 58.995 | 59.548 | 60.147 |
| 4.8 | 60.751 | 61.362 | 61.979 | 62.601 | 63.231 | 63.866 | 64.508 | 65.157 | 65.812 | 66.473 |
| 4.9 | 67.141 | 67.816 | 68.498 | 69.186 | 69.882 | 70.584 | 71.293 | 72.010 | 72.734 | 73.465 |
| 5.0 | 74.203 | 74.949 | 75.702 | 76.463 | 77.232 | 78.008 | 78.792 | 79.584 | 80.384 | 81.192 |
| 5.1 | 82.008 | 82.832 | 83.665 | 84.506 | 85.355 | 86.213 | 87.079 | 87.955 | 88.839 | 89.732 |
| 5.2 | 90.633 | 91.544 | 92.464 | 93.394 | 94.332 | 95.281 | 96.238 | 97.205 | 98.182 | 99.169 |
| 5.3 | 100.17 | 101.17 | 102.19 | 103.22 | 104.25 | 105.30 | 106.36 | 107.43 | 108.51 | 109.60 |
| 5.4 | 110.70 | 111.81 | 112.94 | 114.07 | 115.22 | 116.38 | 117.55 | 118.73 | 119.92 | 121.13 |
| 5.5 | 122.34 | 123.57 | 124.82 | 126.07 | 127.34 | 128.62 | 129.91 | 131.22 | 132.53 | 133.87 |
| 5.6 | 135.21 | 136.57 | 137.94 | 139.33 | 140.73 | 142.14 | 143.57 | 145.02 | 146.47 | 147.95 |
| 5.7 | 149.43 | 150.93 | 152.45 | 153.98 | 155.53 | 157.09 | 158.67 | 160.27 | 161.88 | 163.51 |
| 5.8 | 165.15 | 166.81 | 168.48 | 170.18 | 171.89 | 173.62 | 175.36 | 177.12 | 178.90 | 180.70 |
| 5.9 | 182.52 | 184.35 | 186.20 | 188.08 | 189.97 | 191.88 | 193.80 | 195.75 | 197.72 | 199.71 |

TABLE

# 18b

## HYPERBOLIC FUNCTIONS

## cosh $x$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| .0 | 1.0000 | 1.0001 | 1.0002 | 1.0005 | 1.0008 | 1.0013 | 1.0018 | 1.0025 | 1.0032 | 1.0041 |
| .1 | 1.0050 | 1.0061 | 1.0072 | 1.0085 | 1.0098 | 1.0113 | 1.0128 | 1.0145 | 1.0162 | 1.0181 |
| .2 | 1.0201 | 1.0221 | 1.0243 | 1.0266 | 1.0289 | 1.0314 | 1.0340 | 1.0367 | 1.0395 | 1.0423 |
| .3 | 1.0453 | 1.0484 | 1.0516 | 1.0549 | 1.0584 | 1.0619 | 1.0655 | 1.0692 | 1.0731 | 1.0770 |
| .4 | 1.0811 | 1.0852 | 1.0895 | 1.0939 | 1.0984 | 1.1030 | 1.1077 | 1.1125 | 1.1174 | 1.1225 |
| .5 | 1.1276 | 1.1329 | 1.1383 | 1.1438 | 1.1494 | 1.1551 | 1.1609 | 1.1669 | 1.1730 | 1.1792 |
| .6 | 1.1855 | 1.1919 | 1.1984 | 1.2051 | 1.2119 | 1.2188 | 1.2258 | 1.2330 | 1.2402 | 1.2476 |
| .7 | 1.2552 | 1.2628 | 1.2706 | 1.2785 | 1.2865 | 1.2947 | 1.3030 | 1.3114 | 1.3199 | 1.3286 |
| .8 | 1.3374 | 1.3464 | 1.3555 | 1.3647 | 1.3740 | 1.3835 | 1.3932 | 1.4029 | 1.4128 | 1.4229 |
| .9 | 1.4331 | 1.4434 | 1.4539 | 1.4645 | 1.4753 | 1.4862 | 1.4973 | 1.5085 | 1.5199 | 1.5314 |
| 1.0 | 1.5431 | 1.5549 | 1.5669 | 1.5790 | 1.5913 | 1.6038 | 1.6164 | 1.6292 | 1.6421 | 1.6552 |
| 1.1 | 1.6685 | 1.6820 | 1.6956 | 1.7093 | 1.7233 | 1.7374 | 1.7517 | 1.7662 | 1.7808 | 1.7957 |
| 1.2 | 1.8107 | 1.8258 | 1.8412 | 1.8568 | 1.8725 | 1.8884 | 1.9045 | 1.9208 | 1.9373 | 1.9540 |
| 1.3 | 1.9709 | 1.9880 | 2.0053 | 2.0228 | 2.0404 | 2.0583 | 2.0764 | 2.0947 | 2.1132 | 2.1320 |
| 1.4 | 2.1509 | 2.1700 | 2.1894 | 2.2090 | 2.2288 | 2.2488 | 2.2691 | 2.2896 | 2.3103 | 2.3312 |
| 1.5 | 2.3524 | 2.3738 | 2.3955 | 2.4174 | 2.4395 | 2.4619 | 2.4845 | 2.5073 | 2.5305 | 2.5538 |
| 1.6 | 2.5775 | 2.6013 | 2.6255 | 2.6499 | 2.6746 | 2.6995 | 2.7247 | 2.7502 | 2.7760 | 2.8020 |
| 1.7 | 2.8283 | 2.8549 | 2.8818 | 2.9090 | 2.9364 | 2.9642 | 2.9922 | 3.0206 | 3.0492 | 3.0782 |
| 1.8 | 3.1075 | 3.1371 | 3.1669 | 3.1972 | 3.2277 | 3.2585 | 3.2897 | 3.3212 | 3.3530 | 3.3852 |
| 1.9 | 3.4177 | 3.4506 | 3.4838 | 3.5173 | 3.5512 | 3.5855 | 3.6201 | 3.6551 | 3.6904 | 3.7261 |
| 2.0 | 3.7622 | 3.7987 | 3.8355 | 3.8727 | 3.9103 | 3.9483 | 3.9867 | 4.0255 | 4.0647 | 4.1043 |
| 2.1 | 4.1443 | 4.1847 | 4.2256 | 4.2669 | 4.3085 | 4.3507 | 4.3932 | 4.4362 | 4.4797 | 4.5236 |
| 2.2 | 4.5679 | 4.6127 | 4.6580 | 4.7037 | 4.7499 | 4.7966 | 4.8437 | 4.8914 | 4.9395 | 4.9881 |
| 2.3 | 5.0372 | 5.0868 | 5.1370 | 5.1876 | 5.2388 | 5.2905 | 5.3427 | 5.3954 | 5.4487 | 5.5026 |
| 2.4 | 5.5569 | 5.6119 | 5.6674 | 5.7235 | 5.7801 | 5.8373 | 5.8951 | 5.9535 | 6.0125 | 6.0721 |
| 2.5 | 6.1323 | 6.1931 | 6.2545 | 6.3166 | 6.3793 | 6.4426 | 6.5066 | 6.5712 | 6.6365 | 6.7024 |
| 2.6 | 6.7690 | 6.8363 | 6.9043 | 6.9729 | 7.0423 | 7.1123 | 7.1831 | 7.2546 | 7.3268 | 7.3998 |
| 2.7 | 7.4735 | 7.5479 | 7.6231 | 7.6991 | 7.7758 | 7.8533 | 7.9316 | 8.0106 | 8.0905 | 8.1712 |
| 2.8 | 8.2527 | 8.3351 | 8.4182 | 8.5022 | 8.5871 | 8.6728 | 8.7594 | 8.8469 | 8.9352 | 9.0244 |
| 2.9 | 9.1146 | 9.2056 | 9.2976 | 9.3905 | 9.4844 | 9.5791 | 9.6749 | 9.7716 | 9.8693 | 9.9680 |

**Table 18b**
(continued)

## HYPERBOLIC FUNCTIONS

## cosh $x$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 3.0 | 10.068 | 10.168 | 10.270 | 10.373 | 10.476 | 10.581 | 10.687 | 10.794 | 10.902 | 11.011 |
| 3.1 | 11.121 | 11.233 | 11.345 | 11.459 | 11.574 | 11.689 | 11.806 | 11.925 | 12.044 | 12.165 |
| 3.2 | 12.287 | 12.410 | 12.534 | 12.660 | 12.786 | 12.915 | 13.044 | 13.175 | 13.307 | 13.440 |
| 3.3 | 13.575 | 13.711 | 13.848 | 13.987 | 14.127 | 14.269 | 14.412 | 14.556 | 14.702 | 14.850 |
| 3.4 | 14.999 | 15.149 | 15.301 | 15.455 | 15.610 | 15.766 | 15.924 | 16.084 | 16.245 | 16.408 |
| 3.5 | 16.573 | 16.739 | 16.907 | 17.077 | 17.248 | 17.421 | 17.596 | 17.772 | 17.951 | 18.131 |
| 3.6 | 18.313 | 18.497 | 18.682 | 18.870 | 19.059 | 19.250 | 19.444 | 19.639 | 19.836 | 20.035 |
| 3.7 | 20.236 | 20.439 | 20.644 | 20.852 | 21.061 | 21.272 | 21.486 | 21.702 | 21.919 | 22.139 |
| 3.8 | 22.362 | 22.586 | 22.813 | 23.042 | 23.273 | 23.507 | 23.743 | 23.982 | 24.222 | 24.466 |
| 3.9 | 24.711 | 24.959 | 25.210 | 25.463 | 25.719 | 25.977 | 26.238 | 26.502 | 26.768 | 27.037 |
| 4.0 | 27.308 | 27.583 | 27.860 | 28.139 | 28.422 | 28.707 | 28.996 | 29.287 | 29.581 | 29.878 |
| 4.1 | 30.178 | 30.482 | 30.788 | 31.097 | 31.409 | 31.725 | 32.044 | 32.365 | 32.691 | 33.019 |
| 4.2 | 33.351 | 33.686 | 34.024 | 34.366 | 34.711 | 35.060 | 35.412 | 35.768 | 36.127 | 36.490 |
| 4.3 | 36.857 | 37.227 | 37.601 | 37.979 | 38.360 | 38.746 | 39.135 | 39.528 | 39.925 | 40.326 |
| 4.4 | 40.732 | 41.141 | 41.554 | 41.972 | 42.393 | 42.819 | 43.250 | 43.684 | 44.123 | 44.566 |
| 4.5 | 45.014 | 45.466 | 45.923 | 46.385 | 46.851 | 47.321 | 47.797 | 48.277 | 48.762 | 49.252 |
| 4.6 | 49.747 | 50.247 | 50.752 | 51.262 | 51.777 | 52.297 | 52.823 | 53.354 | 53.890 | 54.431 |
| 4.7 | 54.978 | 55.531 | 56.089 | 56.652 | 57.221 | 57.796 | 58.377 | 58.964 | 59.556 | 60.155 |
| 4.8 | 60.759 | 61.370 | 61.987 | 62.609 | 63.239 | 63.874 | 64.516 | 65.164 | 65.819 | 66.481 |
| 4.9 | 67.149 | 67.823 | 68.505 | 69.193 | 69.889 | 70.591 | 71.300 | 72.017 | 72.741 | 73.472 |
| 5.0 | 74.210 | 74.956 | 75.709 | 76.470 | 77.238 | 78.014 | 78.798 | 79.590 | 80.390 | 81.198 |
| 5.1 | 82.014 | 82.838 | 83.671 | 84.512 | 85.361 | 86.219 | 87.085 | 87.960 | 88.844 | 89.737 |
| 5.2 | 90.639 | 91.550 | 92.470 | 93.399 | 94.338 | 95.286 | 96.243 | 97.211 | 98.188 | 99.174 |
| 5.3 | 100.17 | 101.18 | 102.19 | 103.22 | 104.26 | 105.31 | 106.67 | 107.43 | 108.51 | 109.60 |
| 5.4 | 110.71 | 111.82 | 112.94 | 114.08 | 115.22 | 116.38 | 117.55 | 118.73 | 119.93 | 121.13 |
| 5.5 | 122.35 | 123.58 | 124.82 | 126.07 | 127.34 | 128.62 | 129.91 | 131.22 | 132.54 | 133.87 |
| 5.6 | 135.22 | 136.57 | 137.95 | 139.33 | 140.73 | 142.15 | 143.58 | 145.02 | 146.48 | 147.95 |
| 5.7 | 149.44 | 150.94 | 152.45 | 153.99 | 155.53 | 157.10 | 158.68 | 160.27 | 161.88 | 163.51 |
| 5.8 | 165.15 | 166.81 | 168.49 | 170.18 | 171.89 | 173.62 | 175.36 | 177.13 | 178.91 | 180.70 |
| 5.9 | 182.52 | 184.35 | 186.21 | 188.08 | 189.97 | 191.88 | 193.81 | 195.75 | 197.72 | 199.71 |

TABLE

# 18c

## HYPERBOLIC FUNCTIONS

## tanh $x$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| .0 | .00000 | .01000 | .02000 | .02999 | .03998 | .04996 | .05993 | .06989 | .07983 | .08976 |
| .1 | .09967 | .10956 | .11943 | .12927 | .13909 | .14889 | .15865 | .16838 | .17808 | .18775 |
| .2 | .19738 | .20697 | .21652 | .22603 | .23550 | .24492 | .25430 | .26362 | .27291 | .28213 |
| .3 | .29131 | .30044 | .30951 | .31852 | .32748 | .33638 | .34521 | .35399 | .36271 | .37136 |
| .4 | .37995 | .38847 | .39693 | .40532 | .41364 | .42190 | .43008 | .43820 | .44624 | .45422 |
| .5 | .46212 | .46995 | .47770 | .48538 | .49299 | .50052 | .50798 | .51536 | .52267 | .52990 |
| .6 | .53705 | .54413 | .55113 | .55805 | .56490 | .57167 | .57836 | .58498 | .59152 | .59798 |
| .7 | .60437 | .61068 | .61691 | .62307 | .62915 | .63515 | .64108 | .64693 | .65271 | .65841 |
| .8 | .66404 | .66959 | .67507 | .68048 | .68581 | .69107 | .69626 | .70137 | .70642 | .71139 |
| .9 | .71630 | .72113 | .72590 | .73059 | .73522 | .73978 | .74428 | .74870 | .75307 | .75736 |
| 1.0 | .76159 | .76576 | .76987 | .77391 | .77789 | .78181 | .78566 | .78946 | .79320 | .79688 |
| 1.1 | .80050 | .80406 | .80757 | .81102 | .81441 | .81775 | .82104 | .82427 | .82745 | .83058 |
| 1.2 | .83365 | .83668 | .83965 | .84258 | .84546 | .84828 | .85106 | .85380 | .85648 | .85913 |
| 1.3 | .86172 | .86428 | .86678 | .86925 | .87167 | .87405 | .87639 | .87869 | .88095 | .88317 |
| 1.4 | .88535 | .88749 | .88960 | .89167 | .89370 | .89569 | .89765 | .89958 | .90147 | .90332 |
| 1.5 | .90515 | .90694 | .90870 | .91042 | .91212 | .91379 | .91542 | .91703 | .91860 | .92015 |
| 1.6 | .92167 | .92316 | .92462 | .92606 | .92747 | .92886 | .93022 | .93155 | .93286 | .93415 |
| 1.7 | .93541 | .93665 | .93786 | .93906 | .94023 | .94138 | .94250 | .94361 | .94470 | .94576 |
| 1.8 | .94681 | .94783 | .94884 | .94983 | .95080 | .95175 | .95268 | .95359 | .95449 | .95537 |
| 1.9 | .95624 | .95709 | .95792 | .95873 | .95953 | .96032 | .96109 | .96185 | .96259 | .96331 |
| 2.0 | .96403 | .96473 | .96541 | .96609 | .96675 | .96740 | .96803 | .96865 | .96926 | .96986 |
| 2.1 | .97045 | .97103 | .97159 | .97215 | .97269 | .97323 | .97375 | .97426 | .97477 | .97526 |
| 2.2 | .97574 | .97622 | .97668 | .97714 | .97759 | .97803 | .97846 | .97888 | .97929 | .97970 |
| 2.3 | .98010 | .98049 | .98087 | .98124 | .98161 | .98197 | .98233 | .98267 | .98301 | .98335 |
| 2.4 | .98367 | .98400 | .98431 | .98462 | .98492 | .98522 | .98551 | .98579 | .98607 | .98635 |
| 2.5 | .98661 | .98688 | .98714 | .98739 | .98764 | .98788 | .98812 | .98835 | .98858 | .98881 |
| 2.6 | .98903 | .98924 | .98946 | .98966 | .98987 | .99007 | .99026 | .99045 | .99064 | .99083 |
| 2.7 | .99101 | .99118 | .99136 | .99153 | .99170 | .99186 | .99202 | .99218 | .99233 | .99248 |
| 2.8 | .99263 | .99278 | .99292 | .99306 | .99320 | .99333 | .99346 | .99359 | .99372 | .99384 |
| 2.9 | .99396 | .99408 | .99420 | .99431 | .99443 | .99454 | .99464 | .99475 | .99485 | .99496 |

**Table 18c**
**(continued)**

## HYPERBOLIC FUNCTIONS

## tanh $x$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 3.0 | .99505 | .99515 | .99525 | .99534 | .99543 | .99552 | .99561 | .99570 | .99578 | .99587 |
| 3.1 | .99595 | .99603 | .99611 | .99618 | .99626 | .99633 | .99641 | .99648 | .99655 | .99662 |
| 3.2 | .99668 | .99675 | .99681 | .99688 | .99694 | .99700 | .99706 | .99712 | .99717 | .99723 |
| 3.3 | .99728 | .99734 | .99739 | .99744 | .99749 | .99754 | .99759 | .99764 | .99768 | .99773 |
| 3.4 | .99777 | .99782 | .99786 | .99790 | .99795 | .99799 | .99803 | .99807 | .99810 | .99814 |
| 3.5 | .99818 | .99821 | .99825 | .99828 | .99832 | .99835 | .99838 | .99842 | .99845 | .99848 |
| 3.6 | .99851 | .99853 | .99857 | .99859 | .99862 | .99865 | .99868 | .99870 | .99873 | .99875 |
| 3.7 | .99878 | .99880 | .99883 | .99885 | .99887 | .99889 | .99892 | .99894 | .99896 | .99898 |
| 3.8 | .99900 | .99902 | .99904 | .99906 | .99908 | .99909 | .99911 | .99913 | .99915 | .99916 |
| 3.9 | .99918 | .99920 | .99921 | .99923 | .99924 | .99926 | .99927 | .99929 | .99930 | .99932 |
| 4.0 | .99933 | .99934 | .99936 | .99937 | .99938 | .99939 | .99941 | .99942 | .99943 | .99944 |
| 4.1 | .99945 | .99946 | .99947 | .99948 | .99949 | .99950 | .99951 | .99952 | .99953 | .99954 |
| 4.2 | .99955 | .99956 | .99957 | .99958 | .99958 | .99959 | .99960 | .99961 | .99962 | .99962 |
| 4.3 | .99963 | .99964 | .99965 | .99966 | .99966 | .99967 | .99967 | .99968 | .99969 | .99969 |
| 4.4 | .99970 | .99970 | .99971 | .99972 | .99972 | .99973 | .99973 | .99974 | .99974 | .99975 |
| 4.5 | .99975 | .99976 | .99976 | .99977 | .99977 | .99978 | .99978 | .99979 | .99979 | .99979 |
| 4.6 | .99980 | .99980 | .99981 | .99981 | .99981 | .99982 | .99982 | .99982 | .99983 | .99983 |
| 4.7 | .99983 | .99984 | .99984 | .99984 | .99985 | .99985 | .99985 | .99986 | .99986 | .99986 |
| 4.8 | .99986 | .99987 | .99987 | .99987 | .99987 | .99988 | .99988 | .99988 | .99988 | .99989 |
| 4.9 | .99989 | .99989 | .99990 | .99990 | .99990 | .99990 | .99990 | .99990 | .99991 | .99991 |
| 5.0 | .99991 | .99991 | .99991 | .99991 | .99992 | .99992 | .99992 | .99992 | .99992 | .99992 |
| 5.1 | .99993 | .99993 | .99993 | .99993 | .99993 | .99993 | .99993 | .99994 | .99994 | .99994 |
| 5.2 | .99994 | .99994 | .99994 | .99994 | .99994 | .99994 | .99995 | .99995 | .99995 | .99995 |
| 5.3 | .99995 | .99995 | .99995 | .99995 | .99995 | .99995 | .99996 | .99996 | .99996 | .99996 |
| 5.4 | .99996 | .99996 | .99996 | .99996 | .99996 | .99996 | .99996 | .99996 | .99997 | .99997 |
| 5.5 | .99997 | .99997 | .99997 | .99997 | .99997 | .99997 | .99997 | .99997 | .99997 | .99997 |
| 5.6 | .99997 | .99997 | .99997 | .99997 | .99997 | .99998 | .99998 | .99998 | .99998 | .99998 |
| 5.7 | .99998 | .99998 | .99998 | .99998 | .99998 | .99998 | .99998 | .99998 | .99998 | .99998 |
| 5.8 | .99998 | .99998 | .99998 | .99998 | .99998 | .99998 | .99998 | .99998 | .99998 | .99998 |
| 5.9 | .99998 | .99999 | .99999 | .99999 | .99999 | .99999 | .99999 | .99999 | .99999 | .99999 |

## TABLE
## 19

# FACTORIAL $n$

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$$

| $n$ | $n!$ |
|---|---|
| 0 | 1 (by definition) |
| 1 | 1 |
| 2 | 2 |
| 3 | 6 |
| 4 | 24 |
| 5 | 120 |
| 6 | 720 |
| 7 | 5040 |
| 8 | 40,320 |
| 9 | 362,880 |
| 10 | 3,628,800 |
| 11 | 39,916,800 |
| 12 | 479,001,600 |
| 13 | 6,227,020,800 |
| 14 | 87,178,291,200 |
| 15 | 1,307,674,368,000 |
| 16 | 20,922,789,888,000 |
| 17 | 355,687,428,096,000 |
| 18 | 6,402,373,705,728,000 |
| 19 | 121,645,100,408,832,000 |
| 20 | 2,432,902,008,176,640,000 |
| 21 | 51,090,942,171,709,440,000 |
| 22 | 1,124,000,727,777,607,680,000 |
| 23 | 25,852,016,738,884,976,640,000 |
| 24 | 620,448,401,733,239,439,360,000 |
| 25 | 15,511,210,043,330,985,984,000,000 |
| 26 | 403,291,461,126,605,635,584,000,000 |
| 27 | 10,888,869,450,418,352,160,768,000,000 |
| 28 | 304,888,344,611,713,860,501,504,000,000 |
| 29 | 8,841,761,993,739,701,954,543,616,000,000 |
| 30 | 265,252,859,812,191,058,636,308,480,000,000 |
| 31 | $8.22284 \times 10^{33}$ |
| 32 | $2.63131 \times 10^{35}$ |
| 33 | $8.68332 \times 10^{36}$ |
| 34 | $2.95233 \times 10^{38}$ |
| 35 | $1.03331 \times 10^{40}$ |
| 36 | $3.71993 \times 10^{41}$ |
| 37 | $1.37638 \times 10^{43}$ |
| 38 | $5.23023 \times 10^{44}$ |
| 39 | $2.03979 \times 10^{46}$ |
| 40 | $8.15915 \times 10^{47}$ |
| 41 | $3.34525 \times 10^{49}$ |
| 42 | $1.40501 \times 10^{51}$ |
| 43 | $6.04153 \times 10^{52}$ |
| 44 | $2.65827 \times 10^{54}$ |
| 45 | $1.19622 \times 10^{56}$ |
| 46 | $5.50262 \times 10^{57}$ |
| 47 | $2.58623 \times 10^{59}$ |
| 48 | $1.24139 \times 10^{61}$ |
| 49 | $6.08282 \times 10^{62}$ |
| 50 | $3.04141 \times 10^{64}$ |
| 51 | $1.55112 \times 10^{66}$ |
| 52 | $8.06582 \times 10^{67}$ |
| 53 | $4.27488 \times 10^{69}$ |
| 54 | $2.30844 \times 10^{71}$ |
| 55 | $1.26964 \times 10^{73}$ |
| 56 | $7.10999 \times 10^{74}$ |
| 57 | $4.05269 \times 10^{76}$ |
| 58 | $2.35056 \times 10^{78}$ |
| 59 | $1.38683 \times 10^{80}$ |
| 60 | $8.32099 \times 10^{81}$ |
| 61 | $5.07580 \times 10^{83}$ |
| 62 | $3.14700 \times 10^{85}$ |
| 63 | $1.98261 \times 10^{87}$ |
| 64 | $1.26887 \times 10^{89}$ |
| 65 | $8.24765 \times 10^{90}$ |
| 66 | $5.44345 \times 10^{92}$ |
| 67 | $3.64711 \times 10^{94}$ |
| 68 | $2.48004 \times 10^{96}$ |
| 69 | $1.71122 \times 10^{98}$ |
| 70 | $1.19786 \times 10^{100}$ |
| 71 | $8.50479 \times 10^{101}$ |
| 72 | $6.12345 \times 10^{103}$ |
| 73 | $4.47012 \times 10^{105}$ |
| 74 | $3.30789 \times 10^{107}$ |
| 75 | $2.48091 \times 10^{109}$ |
| 76 | $1.88549 \times 10^{111}$ |
| 77 | $1.45183 \times 10^{113}$ |
| 78 | $1.13243 \times 10^{115}$ |
| 79 | $8.94618 \times 10^{116}$ |
| 80 | $7.15695 \times 10^{118}$ |
| 81 | $5.79713 \times 10^{120}$ |
| 82 | $4.75364 \times 10^{122}$ |
| 83 | $3.94552 \times 10^{124}$ |
| 84 | $3.31424 \times 10^{126}$ |
| 85 | $2.81710 \times 10^{128}$ |
| 86 | $2.42271 \times 10^{130}$ |
| 87 | $2.10776 \times 10^{132}$ |
| 88 | $1.85483 \times 10^{134}$ |
| 89 | $1.65080 \times 10^{136}$ |
| 90 | $1.48572 \times 10^{138}$ |
| 91 | $1.35200 \times 10^{140}$ |
| 92 | $1.24384 \times 10^{142}$ |
| 93 | $1.15677 \times 10^{144}$ |
| 94 | $1.08737 \times 10^{146}$ |
| 95 | $1.03300 \times 10^{148}$ |
| 96 | $9.91678 \times 10^{149}$ |
| 97 | $9.61928 \times 10^{151}$ |
| 98 | $9.42689 \times 10^{153}$ |
| 99 | $9.33262 \times 10^{155}$ |
| 100 | $9.33262 \times 10^{157}$ |

**TABLE**

**20**

## GAMMA FUNCTION

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \quad \text{for } 1 \leqq x \leqq 2$$

[For other values use the formula $\Gamma(x+1) = x\,\Gamma(x)$]

| $x$ | $\Gamma(x)$ | $x$ | $\Gamma(x)$ |
|---|---|---|---|
| 1.00 | 1.00000 | 1.50 | .88623 |
| 1.01 | .99433 | 1.51 | .88659 |
| 1.02 | .98884 | 1.52 | .88704 |
| 1.03 | .98355 | 1.53 | .88757 |
| 1.04 | .97844 | 1.54 | .88818 |
| 1.05 | .97350 | 1.55 | .88887 |
| 1.06 | .96874 | 1.56 | .88964 |
| 1.07 | .96415 | 1.57 | .89049 |
| 1.08 | .95973 | 1.58 | .89142 |
| 1.09 | .95546 | 1.59 | .89243 |
| 1.10 | .95135 | 1.60 | .89352 |
| 1.11 | .94740 | 1.61 | .89468 |
| 1.12 | .94359 | 1.62 | .89592 |
| 1.13 | .93993 | 1.63 | .89724 |
| 1.14 | .93642 | 1.64 | .89864 |
| 1.15 | .93304 | 1.65 | .90012 |
| 1.16 | .92980 | 1.66 | .90167 |
| 1.17 | .92670 | 1.67 | .90330 |
| 1.18 | .92373 | 1.68 | .90500 |
| 1.19 | .92089 | 1.69 | .90678 |
| 1.20 | .91817 | 1.70 | .90864 |
| 1.21 | .91558 | 1.71 | .91057 |
| 1.22 | .91311 | 1.72 | .91258 |
| 1.23 | .91075 | 1.73 | .91467 |
| 1.24 | .90852 | 1.74 | .91683 |
| 1.25 | .90640 | 1.75 | .91906 |
| 1.26 | .90440 | 1.76 | .92137 |
| 1.27 | .90250 | 1.77 | .92376 |
| 1.28 | .90072 | 1.78 | .92623 |
| 1.29 | .89904 | 1.79 | .92877 |
| 1.30 | .89747 | 1.80 | .93138 |
| 1.31 | .89600 | 1.81 | .93408 |
| 1.32 | .89464 | 1.82 | .93685 |
| 1.33 | .89338 | 1.83 | .93969 |
| 1.34 | .89222 | 1.84 | .94261 |
| 1.35 | .89115 | 1.85 | .94561 |
| 1.36 | .89018 | 1.86 | .94869 |
| 1.37 | .88931 | 1.87 | .95184 |
| 1.38 | .88854 | 1.88 | .95507 |
| 1.39 | .88785 | 1.89 | .95838 |
| 1.40 | .88726 | 1.90 | .96177 |
| 1.41 | .88676 | 1.91 | .96523 |
| 1.42 | .88636 | 1.92 | .96877 |
| 1.43 | .88604 | 1.93 | .97240 |
| 1.44 | .88581 | 1.94 | .97610 |
| 1.45 | .88566 | 1.95 | .97988 |
| 1.46 | .88560 | 1.96 | .98374 |
| 1.47 | .88563 | 1.97 | .98768 |
| 1.48 | .88575 | 1.98 | .99171 |
| 1.49 | .88595 | 1.99 | .99581 |
| 1.50 | .88623 | 2.00 | 1.00000 |

TABLE

# 21

## BINOMIAL COEFFICIENTS

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \binom{n}{n-k}, \quad 0! = 1$$

Note that each number is the sum of two numbers in the row above; one of these numbers is in the same column and the other is in the preceding column [e.g. 56 = 35 + 21]. The arrangement is often called *Pascal's triangle* [see 3.6, page 4].

| $n$ \ $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | | | | | | | | |
| 2 | 1 | 2 | 1 | | | | | | | |
| 3 | 1 | 3 | 3 | 1 | | | | | | |
| 4 | 1 | 4 | 6 | 4 | 1 | | | | | |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 | | | | |
| 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 | | | |
| 7 | 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 | | |
| 8 | 1 | 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 | |
| 9 | 1 | 9 | 36 | 84 | 126 | 126 | 84 | 36 | 9 | 1 |
| 10 | 1 | 10 | 45 | 120 | 210 | 252 | 210 | 120 | 45 | 10 |
| 11 | 1 | 11 | 55 | 165 | 330 | 462 | 462 | 330 | 165 | 55 |
| 12 | 1 | 12 | 66 | 220 | 495 | 792 | 924 | 792 | 495 | 220 |
| 13 | 1 | 13 | 78 | 286 | 715 | 1287 | 1716 | 1716 | 1287 | 715 |
| 14 | 1 | 14 | 91 | 364 | 1001 | 2002 | 3003 | 3432 | 3003 | 2002 |
| 15 | 1 | 15 | 105 | 455 | 1365 | 3003 | 5005 | 6435 | 6435 | 5005 |
| 16 | 1 | 16 | 120 | 560 | 1820 | 4368 | 8008 | 11440 | 12870 | 11440 |
| 17 | 1 | 17 | 136 | 680 | 2380 | 6188 | 12376 | 19448 | 24310 | 24310 |
| 18 | 1 | 18 | 153 | 816 | 3060 | 8568 | 18564 | 31824 | 43758 | 48620 |
| 19 | 1 | 19 | 171 | 969 | 3876 | 11628 | 27132 | 50388 | 75582 | 92378 |
| 20 | 1 | 20 | 190 | 1140 | 4845 | 15504 | 38760 | 77520 | 125970 | 167960 |
| 21 | 1 | 21 | 210 | 1330 | 5985 | 20349 | 54264 | 116280 | 203490 | 293930 |
| 22 | 1 | 22 | 231 | 1540 | 7315 | 26334 | 74613 | 170544 | 319770 | 497420 |
| 23 | 1 | 23 | 253 | 1771 | 8855 | 33649 | 100947 | 245157 | 490314 | 817190 |
| 24 | 1 | 24 | 276 | 2024 | 10626 | 42504 | 134596 | 346104 | 735471 | 1307504 |
| 25 | 1 | 25 | 300 | 2300 | 12650 | 53130 | 177100 | 480700 | 1081575 | 2042975 |
| 26 | 1 | 26 | 325 | 2600 | 14950 | 65780 | 230230 | 657800 | 1562275 | 3124550 |
| 27 | 1 | 27 | 351 | 2925 | 17550 | 80730 | 296010 | 888030 | 2220075 | 4686825 |
| 28 | 1 | 28 | 378 | 3276 | 20475 | 98280 | 376740 | 1184040 | 3108105 | 6906900 |
| 29 | 1 | 29 | 406 | 3654 | 23751 | 118755 | 475020 | 1560780 | 4292145 | 10015005 |
| 30 | 1 | 30 | 435 | 4060 | 27405 | 142506 | 593775 | 2035800 | 5852925 | 14307150 |

**Table 21**
**(continued)**

## BINOMIAL COEFFICIENTS

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \binom{n}{n-k}, \quad 0! = 1$$

| $n$ \ $k$ | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|
| 10 | 1 | | | | | |
| 11 | 11 | 1 | | | | |
| 12 | 66 | 12 | 1 | | | |
| 13 | 286 | 78 | 13 | 1 | | |
| 14 | 1001 | 364 | 91 | 14 | 1 | |
| 15 | 3003 | 1365 | 455 | 105 | 15 | 1 |
| 16 | 8008 | 4368 | 1820 | 560 | 120 | 16 |
| 17 | 19448 | 12376 | 6188 | 2380 | 680 | 136 |
| 18 | 43758 | 31824 | 18564 | 8568 | 3060 | 816 |
| 19 | 92378 | 75582 | 50388 | 27132 | 11628 | 3876 |
| 20 | 184756 | 167960 | 125970 | 77520 | 38760 | 15504 |
| 21 | 352716 | 352716 | 293930 | 203490 | 116280 | 54264 |
| 22 | 646646 | 705432 | 646646 | 497420 | 319770 | 170544 |
| 23 | 1144066 | 1352078 | 1352078 | 1144066 | 817190 | 490314 |
| 24 | 1961256 | 2496144 | 2704156 | 2496144 | 1961256 | 1307504 |
| 25 | 3268760 | 4457400 | 5200300 | 5200300 | 4457400 | 3268760 |
| 26 | 5311735 | 7726160 | 9657700 | 10400600 | 9657700 | 7726160 |
| 27 | 8436285 | 13037895 | 17383860 | 20058300 | 20058300 | 17383860 |
| 28 | 13123110 | 21474180 | 30421755 | 37442160 | 40116600 | 37442160 |
| 29 | 20030010 | 34597290 | 51895935 | 67863915 | 77558760 | 77558760 |
| 30 | 30045015 | 54627300 | 86493225 | 119759850 | 145422675 | 155117520 |

For $k > 15$ use the fact that $\binom{n}{k} = \binom{n}{n-k}$.

**TABLE**

**22**

## SQUARES, CUBES, ROOTS AND RECIPROCALS

| $n$ | $n^2$ | $n^3$ | $\sqrt{n}$ | $\sqrt{10n}$ | $\sqrt[3]{n}$ | $\sqrt[3]{10n}$ | $\sqrt[3]{100n}$ | $1/n$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1.000 000 | 3.162 278 | 1.000 000 | 2.154 435 | 4.641 589 | 1.000 000 |
| 2 | 4 | 8 | 1.414 214 | 4.472 136 | 1.259 921 | 2.714 418 | 5.848 035 | .500 000 |
| 3 | 9 | 27 | 1.732 051 | 5.477 226 | 1.442 250 | 3.107 233 | 6.694 330 | .333 333 |
| 4 | 16 | 64 | 2.000 000 | 6.324 555 | 1.587 401 | 3.419 952 | 7.368 063 | .250 000 |
| 5 | 25 | 125 | 2.236 068 | 7.071 068 | 1.709 976 | 3.684 031 | 7.937 005 | .200 000 |
| 6 | 36 | 216 | 2.449 490 | 7.745 967 | 1.817 121 | 3.914 868 | 8.434 327 | .166 667 |
| 7 | 49 | 343 | 2.645 751 | 8.366 600 | 1.912 931 | 4.121 285 | 8.879 040 | .142 857 |
| 8 | 64 | 512 | 2.828 427 | 8.944 272 | 2.000 000 | 4.308 869 | 9.283 178 | .125 000 |
| 9 | 81 | 729 | 3.000 000 | 9.486 833 | 2.080 084 | 4.481 405 | 9.654 894 | .111 111 |
| 10 | 100 | 1 000 | 3.162 278 | 10.000 00 | 2.154 435 | 4.641 589 | 10.000 00 | .100 000 |
| 11 | 121 | 1 331 | 3.316 625 | 10.488 09 | 2.223 980 | 4.791 420 | 10.322 80 | .090 909 |
| 12 | 144 | 1 728 | 3.464 102 | 10.954 45 | 2.289 428 | 4.932 424 | 10.626 59 | .083 333 |
| 13 | 169 | 2 197 | 3.605 551 | 11.401 75 | 2.351 335 | 5.065 797 | 10.913 93 | .076 923 |
| 14 | 196 | 2 744 | 3.741 657 | 11.832 16 | 2.410 142 | 5.192 494 | 11.186 89 | .071 429 |
| 15 | 225 | 3 375 | 3.872 983 | 12.247 45 | 2.466 212 | 5.313 293 | 11.447 14 | .066 667 |
| 16 | 256 | 4 096 | 4.000 000 | 12.649 11 | 2.519 842 | 5.428 835 | 11.696 07 | .062 500 |
| 17 | 289 | 4 913 | 4.123 106 | 13.038 40 | 2.571 282 | 5.539 658 | 11.934 83 | .058 824 |
| 18 | 324 | 5 832 | 4.242 641 | 13.416 41 | 2.620 741 | 5.646 216 | 12.164 40 | .055 556 |
| 19 | 361 | 6 859 | 4.358 899 | 13.784 05 | 2.668 402 | 5.748 897 | 12.385 62 | .052 632 |
| 20 | 400 | 8 000 | 4.472 136 | 14.142 14 | 2.714 418 | 5.848 035 | 12.599 21 | .050 000 |
| 21 | 441 | 9 261 | 4.582 576 | 14.491 38 | 2.758 924 | 5.943 922 | 12.805 79 | .047 619 |
| 22 | 484 | 10 648 | 4.690 416 | 14.832 40 | 2.802 039 | 6.036 811 | 13.005 91 | .045 455 |
| 23 | 529 | 12 167 | 4.795 832 | 15.165 75 | 2.843 867 | 6.126 926 | 13.200 06 | .043 478 |
| 24 | 576 | 13 824 | 4.898 979 | 15.491 93 | 2.884 499 | 6.214 465 | 13.388 66 | .041 667 |
| 25 | 625 | 15 625 | 5.000 000 | 15.811 39 | 2.924 018 | 6.299 605 | 13.572 09 | .040 000 |
| 26 | 676 | 17 576 | 5.099 020 | 16.124 52 | 2.962 496 | 6.382 504 | 13.750 69 | .038 462 |
| 27 | 729 | 19 683 | 5.196 152 | 16.431 68 | 3.000 000 | 6.463 304 | 13.924 77 | .037 037 |
| 28 | 784 | 21 952 | 5.291 503 | 16.733 20 | 3.036 589 | 6.542 133 | 14.094 60 | .035 714 |
| 29 | 841 | 24 389 | 5.385 165 | 17.029 39 | 3.072 317 | 6.619 106 | 14.260 43 | .034 483 |
| 30 | 900 | 27 000 | 5.477 226 | 17.320 51 | 3.107 233 | 6.694 330 | 14.422 50 | .033 333 |
| 31 | 961 | 29 791 | 5.567 764 | 17.606 82 | 3.141 381 | 6.767 899 | 14.581 00 | .032 258 |
| 32 | 1 024 | 32 768 | 5.656 854 | 17.888 54 | 3.174 802 | 6.839 904 | 14.736 13 | .031 250 |
| 33 | 1 089 | 35 937 | 5.744 563 | 18.165 90 | 3.207 534 | 6.910 423 | 14.888 06 | .030 303 |
| 34 | 1 156 | 39 304 | 5.830 952 | 18.439 09 | 3.239 612 | 6.979 532 | 15.036 95 | .029 412 |
| 35 | 1 225 | 42 875 | 5.916 080 | 18.708 29 | 3.271 066 | 7.047 299 | 15.182 94 | .028 571 |
| 36 | 1 296 | 46 656 | 6.000 000 | 18.973 67 | 3.301 927 | 7.113 787 | 15.326 19 | .027 778 |
| 37 | 1 369 | 50 653 | 6.082 763 | 19.235 38 | 3.332 222 | 7.179 054 | 15.466 80 | .027 027 |
| 38 | 1 444 | 54 872 | 6.164 414 | 19.493 59 | 3.361 975 | 7.243 156 | 15.604 91 | .026 316 |
| 39 | 1 521 | 59 319 | 6.244 998 | 19.748 42 | 3.391 211 | 7.306 144 | 15.740 61 | .025 641 |
| 40 | 1 600 | 64 000 | 6.324 555 | 20.000 00 | 3.419 952 | 7.368 063 | 15.874 01 | .025 000 |
| 41 | 1 681 | 68 921 | 6.403 124 | 20.248 46 | 3.448 217 | 7.428 959 | 16.005 21 | .024 390 |
| 42 | 1 764 | 74 088 | 6.480 741 | 20.493 90 | 3.476 027 | 7.488 872 | 16.134 29 | .023 810 |
| 43 | 1 849 | 79 507 | 6.557 439 | 20.736 44 | 3.503 398 | 7.547 842 | 16.261 33 | .023 256 |
| 44 | 1 936 | 85 184 | 6.633 250 | 20.976 18 | 3.530 348 | 7.605 905 | 16.386 43 | .022 727 |
| 45 | 2 025 | 91 125 | 6.708 204 | 21.213 20 | 3.556 893 | 7.663 094 | 16.509 64 | .022 222 |
| 46 | 2 116 | 97 336 | 6.782 330 | 21.447 61 | 3.583 048 | 7.719 443 | 16.631 03 | .021 739 |
| 47 | 2 209 | 103 823 | 6.855 655 | 21.679 48 | 3.608 826 | 7.774 980 | 16.750 69 | .021 277 |
| 48 | 2 304 | 110 592 | 6.928 203 | 21.908 90 | 3.634 241 | 7.829 735 | 16.868 65 | .020 833 |
| 49 | 2 401 | 117 649 | 7.000 000 | 22.135 94 | 3.659 306 | 7.883 735 | 16.984 99 | .020 408 |
| 50 | 2 500 | 125 000 | 7.071 068 | 22.360 68 | 3.684 031 | 7.937 005 | 17.099 76 | .020 000 |

## Table 22 (continued)

## SQUARES, CUBES, ROOTS AND RECIPROCALS

| $n$ | $n^2$ | $n^3$ | $\sqrt{n}$ | $\sqrt{10n}$ | $\sqrt[3]{n}$ | $\sqrt[3]{10n}$ | $\sqrt[3]{100n}$ | $1/n$ |
|---|---|---|---|---|---|---|---|---|
| 50 | 2 500 | 125 000 | 7.071 068 | 22.360 68 | 3.684 031 | 7.937 005 | 17.099 76 | .020 000 |
| 51 | 2 601 | 132 651 | 7.141 428 | 22.583 18 | 3.708 430 | 7.989 570 | 17.213 01 | .019 608 |
| 52 | 2 704 | 140 608 | 7.211 103 | 22.803 51 | 3.732 511 | 8.041 452 | 17.324 78 | .019 231 |
| 53 | 2 809 | 148 877 | 7.280 110 | 23.021 73 | 3.756 286 | 8.092 672 | 17.435 13 | .018 868 |
| 54 | 2 916 | 157 464 | 7.348 469 | 23.237 90 | 3.779 763 | 8.143 253 | 17.544 11 | .018 519 |
| 55 | 3 025 | 166 375 | 7.416 198 | 23.452 08 | 3.802 952 | 8.193 213 | 17.651 74 | .018 182 |
| 56 | 3 136 | 175 616 | 7.483 315 | 23.664 32 | 3.825 862 | 8.242 571 | 17.758 08 | .017 857 |
| 57 | 3 249 | 185 193 | 7.549 834 | 23.874 67 | 3.848 501 | 8.291 344 | 17.863 16 | .017 544 |
| 58 | 3 364 | 195 112 | 7.615 773 | 24.083 19 | 3.870 877 | 8.339 551 | 17.967 02 | .017 241 |
| 59 | 3 481 | 205 379 | 7.681 146 | 24.289 92 | 3.892 996 | 8.387 207 | 18.069 69 | .016 949 |
| 60 | 3 600 | 216 000 | 7.745 967 | 24.494 90 | 3.914 868 | 8.434 327 | 18.171 21 | .016 667 |
| 61 | 3 721 | 226 981 | 7.810 250 | 24.698 18 | 3.936 497 | 8.480 926 | 18.271 60 | .016 393 |
| 62 | 3 844 | 238 328 | 7.874 008 | 24.899 80 | 3.957 892 | 8.527 019 | 18.370 91 | .016 129 |
| 63 | 3 969 | 250 047 | 7.937 254 | 25.099 80 | 3.979 057 | 8.572 619 | 18.469 15 | .015 873 |
| 64 | 4 096 | 262 144 | 8.000 000 | 25.298 22 | 4.000 000 | 8.617 739 | 18.566 36 | .015 625 |
| 65 | 4 225 | 274 625 | 8.062 258 | 25.495 10 | 4.020 726 | 8.662 391 | 18.662 56 | .015 385 |
| 66 | 4 356 | 287 496 | 8.124 038 | 25.690 47 | 4.041 240 | 8.706 588 | 18.757 77 | .015 152 |
| 67 | 4 489 | 300 763 | 8.185 353 | 25.884 36 | 4.061 548 | 8.750 340 | 18.852 04 | .014 925 |
| 68 | 4 624 | 314 432 | 8.246 211 | 26.076 81 | 4.081 655 | 8.793 659 | 18.945 36 | .014 706 |
| 69 | 4 761 | 328 509 | 8.306 624 | 26.267 85 | 4.101 566 | 8.836 556 | 19.037 78 | .014 493 |
| 70 | 4 900 | 343 000 | 8.366 600 | 26.457 51 | 4.121 285 | 8.879 040 | 19.129 31 | .014 286 |
| 71 | 5 041 | 357 911 | 8.426 150 | 26.645 83 | 4.140 818 | 8.921 121 | 19.219 97 | .014 085 |
| 72 | 5 184 | 373 248 | 8.485 281 | 26.832 82 | 4.160 168 | 8.962 809 | 19.309 79 | .013 889 |
| 73 | 5 329 | 389 017 | 8.544 004 | 27.018 51 | 4.179 339 | 9.004 113 | 19.398 77 | .013 699 |
| 74 | 5 476 | 405 224 | 8.602 325 | 27.202 94 | 4.198 336 | 9.045 042 | 19.486 95 | .013 514 |
| 75 | 5 625 | 421 875 | 8.660 254 | 27.386 13 | 4.217 163 | 9.085 603 | 19.574 34 | .013 333 |
| 76 | 5 776 | 438 976 | 8.717 798 | 27.568 10 | 4.235 824 | 9.125 805 | 19.660 95 | .013 158 |
| 77 | 5 929 | 456 533 | 8.774 964 | 27.748 87 | 4.254 321 | 9.165 656 | 19.746 81 | .012 987 |
| 78 | 6 084 | 474 552 | 8.831 761 | 27.928 48 | 4.272 659 | 9.205 164 | 19.831 92 | .012 821 |
| 79 | 6 241 | 493 039 | 8.888 194 | 28.106 94 | 4.290 840 | 9.244 335 | 19.916 32 | .012 658 |
| 80 | 6 400 | 512 000 | 8.944 272 | 28.284 27 | 4.308 869 | 9.283 178 | 20.000 00 | .012 500 |
| 81 | 6 561 | 531 441 | 9.000 000 | 28.460 50 | 4.326 749 | 9.321 698 | 20.082 99 | .012 346 |
| 82 | 6 724 | 551 368 | 9.055 385 | 28.635 64 | 4.344 481 | 9.359 902 | 20.165 30 | .012 195 |
| 83 | 6 889 | 571 787 | 9.110 434 | 28.809 72 | 4.362 071 | 9.397 796 | 20.246 94 | .012 048 |
| 84 | 7 056 | 592 704 | 9.165 151 | 28.982 75 | 4.379 519 | 9.435 388 | 20.327 93 | .011 905 |
| 85 | 7 225 | 614 125 | 9.219 544 | 29.154 76 | 4.396 830 | 9.472 682 | 20.408 28 | .011 765 |
| 86 | 7 396 | 636 056 | 9.273 618 | 29.325 76 | 4.414 005 | 9.509 685 | 20.488 00 | .011 628 |
| 87 | 7 569 | 658 503 | 9.327 379 | 29.495 76 | 4.431 048 | 9.546 403 | 20.567 10 | .011 494 |
| 88 | 7 744 | 681 472 | 9.380 832 | 29.664 79 | 4.447 960 | 9.582 840 | 20.645 60 | .011 364 |
| 89 | 7 921 | 704 969 | 9.433 981 | 29.832 87 | 4.464 745 | 9.619 002 | 20.723 51 | .011 236 |
| 90 | 8 100 | 729 000 | 9.486 833 | 30.000 00 | 4.481 405 | 9.654 894 | 20.800 84 | .011 111 |
| 91 | 8 281 | 753 571 | 9.539 392 | 30.166 21 | 4.497 941 | 9.690 521 | 20.877 59 | .010 989 |
| 92 | 8 464 | 778 688 | 9.591 663 | 30.331 50 | 4.514 357 | 9.725 888 | 20.953 79 | .010 870 |
| 93 | 8 649 | 804 357 | 9.643 651 | 30.495 90 | 4.530 655 | 9.761 000 | 21.029 44 | .010 753 |
| 94 | 8 836 | 830 584 | 9.695 360 | 30.659 42 | 4.546 836 | 9.795 861 | 21.104 54 | .010 638 |
| 95 | 9 025 | 857 375 | 9.746 794 | 30.822 07 | 4.562 903 | 9.830 476 | 21.179 12 | .010 526 |
| 96 | 9 216 | 884 736 | 9.797 959 | 30.983 87 | 4.578 857 | 9.864 848 | 21.253 17 | .010 417 |
| 97 | 9 409 | 912 673 | 9.848 858 | 31.144 82 | 4.594 701 | 9.898 983 | 21.326 71 | .010 309 |
| 98 | 9 604 | 941 192 | 9.899 495 | 31.304 95 | 4.610 436 | 9.932 884 | 21.399 75 | .010 204 |
| 99 | 9 801 | 970 299 | 9.949 874 | 31.464 27 | 4.626 065 | 9.966 555 | 21.472 29 | .010 101 |
| 100 | 10 000 | 1 000 000 | 10.00 000 | 31.622 78 | 4.641 589 | 10.00 000 | 21.544 35 | .010 000 |

**TABLE**
# 23

## COMPOUND AMOUNT: $(1+r)^n$

If a principal $P$ is deposited at interest rate $r$ (in decimals) compounded annually, then at the end of $n$ years the accumulated amount $A = P(1+r)^n$.

| $n$ \ $r$ | 1% | $1\frac{1}{4}$% | $1\frac{1}{2}$% | 2% | $2\frac{1}{2}$% | 3% | 4% | 5% | 6% |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1.0100 | 1.0125 | 1.0150 | 1.0200 | 1.0250 | 1.0300 | 1.0400 | 1.0500 | 1.0600 |
| 2 | 1.0201 | 1.0252 | 1.0302 | 1.0404 | 1.0506 | 1.0609 | 1.0816 | 1.1025 | 1.1236 |
| 3 | 1.0303 | 1.0380 | 1.0457 | 1.0612 | 1.0769 | 1.0927 | 1.1249 | 1.1576 | 1.1910 |
| 4 | 1.0406 | 1.0509 | 1.0614 | 1.0824 | 1.1038 | 1.1255 | 1.1699 | 1.2155 | 1.2635 |
| 5 | 1.0510 | 1.0641 | 1.0773 | 1.1041 | 1.1314 | 1.1593 | 1.2167 | 1.2763 | 1.3382 |
| 6 | 1.0615 | 1.0774 | 1.0934 | 1.1262 | 1.1597 | 1.1941 | 1.2653 | 1.3401 | 1.4185 |
| 7 | 1.0721 | 1.0909 | 1.1098 | 1.1487 | 1.1887 | 1.2299 | 1.3159 | 1.4071 | 1.5036 |
| 8 | 1.0829 | 1.1045 | 1.1265 | 1.1717 | 1.2184 | 1.2668 | 1.3688 | 1.4775 | 1.5938 |
| 9 | 1.0937 | 1.1183 | 1.1434 | 1.1951 | 1.2489 | 1.3048 | 1.4233 | 1.5513 | 1.6895 |
| 10 | 1.1046 | 1.1323 | 1.1605 | 1.2190 | 1.2801 | 1.3439 | 1.4802 | 1.6289 | 1.7908 |
| 11 | 1.1157 | 1.1464 | 1.1779 | 1.2434 | 1.3121 | 1.3842 | 1.5395 | 1.7103 | 1.8983 |
| 12 | 1.1268 | 1.1608 | 1.1956 | 1.2682 | 1.3449 | 1.4258 | 1.6010 | 1.7959 | 2.0122 |
| 13 | 1.1381 | 1.1753 | 1.2136 | 1.2936 | 1.3785 | 1.4685 | 1.6651 | 1.8856 | 2.1329 |
| 14 | 1.1495 | 1.1900 | 1.2318 | 1.3195 | 1.4130 | 1.5126 | 1.7317 | 1.9799 | 2.2609 |
| 15 | 1.1610 | 1.2048 | 1.2502 | 1.3459 | 1.4483 | 1.5580 | 1.8009 | 2.0789 | 2.3966 |
| 16 | 1.1726 | 1.2199 | 1.2690 | 1.3728 | 1.4845 | 1.6047 | 1.8730 | 2.1829 | 2.5404 |
| 17 | 1.1843 | 1.2351 | 1.2880 | 1.4002 | 1.5216 | 1.6528 | 1.9479 | 2.2920 | 2.6928 |
| 18 | 1.1961 | 1.2506 | 1.3073 | 1.4282 | 1.5597 | 1.7024 | 2.0258 | 2.4066 | 2.8543 |
| 19 | 1.2081 | 1.2662 | 1.3270 | 1.4568 | 1.5987 | 1.7535 | 2.1068 | 2.5270 | 3.0256 |
| 20 | 1.2202 | 1.2820 | 1.3469 | 1.4859 | 1.6386 | 1.8061 | 2.1911 | 2.6533 | 3.2071 |
| 21 | 1.2324 | 1.2981 | 1.3671 | 1.5157 | 1.6796 | 1.8603 | 2.2788 | 2.7860 | 3.3996 |
| 22 | 1.2447 | 1.3143 | 1.3876 | 1.5460 | 1.7216 | 1.9161 | 2.3699 | 2.9253 | 3.6035 |
| 23 | 1.2572 | 1.3307 | 1.4084 | 1.5769 | 1.7646 | 1.9736 | 2.4647 | 3.0715 | 3.8197 |
| 24 | 1.2697 | 1.3474 | 1.4295 | 1.6084 | 1.8087 | 2.0328 | 2.5633 | 3.2251 | 4.0489 |
| 25 | 1.2824 | 1.3642 | 1.4509 | 1.6406 | 1.8539 | 2.0938 | 2.6658 | 3.3864 | 4.2919 |
| 26 | 1.2953 | 1.3812 | 1.4727 | 1.6734 | 1.9003 | 2.1566 | 2.7725 | 3.5557 | 4.5494 |
| 27 | 1.3082 | 1.3985 | 1.4948 | 1.7069 | 1.9478 | 2.2213 | 2.8834 | 3.7335 | 4.8223 |
| 28 | 1.3213 | 1.4160 | 1.5172 | 1.7410 | 1.9965 | 2.2879 | 2.9987 | 3.9201 | 5.1117 |
| 29 | 1.3345 | 1.4337 | 1.5400 | 1.7758 | 2.0464 | 2.3566 | 3.1187 | 4.1161 | 5.4184 |
| 30 | 1.3478 | 1.4516 | 1.5631 | 1.8114 | 2.0976 | 2.4273 | 3.2434 | 4.3219 | 5.7435 |
| 31 | 1.3613 | 1.4698 | 1.5865 | 1.8476 | 2.1500 | 2.5001 | 3.3731 | 4.5380 | 6.0881 |
| 32 | 1.3749 | 1.4881 | 1.6103 | 1.8845 | 2.2038 | 2.5751 | 3.5081 | 4.7649 | 6.4534 |
| 33 | 1.3887 | 1.5067 | 1.6345 | 1.9222 | 2.2589 | 2.6523 | 3.6484 | 5.0032 | 6.8406 |
| 34 | 1.4026 | 1.5256 | 1.6590 | 1.9607 | 2.3153 | 2.7319 | 3.7943 | 5.2533 | 7.2510 |
| 35 | 1.4166 | 1.5446 | 1.6839 | 1.9999 | 2.3732 | 2.8139 | 3.9461 | 5.5160 | 7.6861 |
| 36 | 1.4308 | 1.5639 | 1.7091 | 2.0399 | 2.4325 | 2.8983 | 4.1039 | 5.7918 | 8.1473 |
| 37 | 1.4451 | 1.5835 | 1.7348 | 2.0807 | 2.4933 | 2.9852 | 4.2681 | 6.0814 | 8.6361 |
| 38 | 1.4595 | 1.6033 | 1.7608 | 2.1223 | 2.5557 | 3.0748 | 4.4388 | 6.3855 | 9.1543 |
| 39 | 1.4741 | 1.6233 | 1.7872 | 2.1647 | 2.6196 | 3.1670 | 4.6164 | 6.7048 | 9.7035 |
| 40 | 1.4889 | 1.6436 | 1.8140 | 2.2080 | 2.6851 | 3.2620 | 4.8010 | 7.0400 | 10.2857 |
| 41 | 1.5038 | 1.6642 | 1.8412 | 2.2522 | 2.7522 | 3.3599 | 4.9931 | 7.3920 | 10.9029 |
| 42 | 1.5188 | 1.6850 | 1.8688 | 2.2972 | 2.8210 | 3.4607 | 5.1928 | 7.7616 | 11.5570 |
| 43 | 1.5340 | 1.7060 | 1.8969 | 2.3432 | 2.8915 | 3.5645 | 5.4005 | 8.1497 | 12.2505 |
| 44 | 1.5493 | 1.7274 | 1.9253 | 2.3901 | 2.9638 | 3.6715 | 5.6165 | 8.5572 | 12.9855 |
| 45 | 1.5648 | 1.7489 | 1.9542 | 2.4379 | 3.0379 | 3.7816 | 5.8412 | 8.9850 | 13.7646 |
| 46 | 1.5805 | 1.7708 | 1.9835 | 2.4866 | 3.1139 | 3.8950 | 6.0748 | 9.4343 | 14.5905 |
| 47 | 1.5963 | 1.7929 | 2.0133 | 2.5363 | 3.1917 | 4.0119 | 6.3178 | 9.9060 | 15.4659 |
| 48 | 1.6122 | 1.8154 | 2.0435 | 2.5871 | 3.2715 | 4.1323 | 6.5705 | 10.4013 | 16.3939 |
| 49 | 1.6283 | 1.8380 | 2.0741 | 2.6388 | 3.3533 | 4.2562 | 6.8333 | 10.9213 | 17.3775 |
| 50 | 1.6446 | 1.8610 | 2.1052 | 2.6916 | 3.4371 | 4.3839 | 7.1067 | 11.4674 | 18.4202 |

TABLE

# 24

## PRESENT VALUE OF AN AMOUNT: $(1+r)^{-n}$

The present value $P$ which will amount to $A$ in $n$ years at an interest rate of $r$ (in decimals) compounded annually is $P = A(1+r)^{n}$.

| $r$<br>$n$ | 1% | $1\frac{1}{4}$% | $1\frac{1}{2}$% | 2% | $2\frac{1}{2}$% | 3% | 4% | 5% | 6% |
|---|---|---|---|---|---|---|---|---|---|
| 1 | .99010 | .98765 | .98522 | .98039 | .97561 | .97087 | .96154 | .95238 | .94340 |
| 2 | .98030 | .97546 | .97066 | .96117 | .95181 | .94260 | .92456 | .90703 | .89000 |
| 3 | .97059 | .96342 | .95632 | .94232 | .92860 | .91514 | .88900 | .86384 | .83962 |
| 4 | .96098 | .95152 | .94218 | .92385 | .90595 | .88849 | .85480 | .82270 | .79209 |
| 5 | .95147 | .93978 | .92826 | .90573 | .88385 | .86261 | .82193 | .78353 | .74726 |
| 6 | .94205 | .92817 | .91454 | .88797 | .86230 | .83748 | .79031 | .74622 | .70496 |
| 7 | .93272 | .91672 | .90103 | .87056 | .84127 | .81309 | .75992 | .71068 | .66506 |
| 8 | .92348 | .90540 | .88771 | .85349 | .82075 | .78941 | .73069 | .67684 | .62741 |
| 9 | .91434 | .89422 | .87459 | .83676 | .80073 | .76642 | .70259 | .64461 | .59190 |
| 10 | .90529 | .88318 | .86167 | .82035 | .78120 | .74409 | .67556 | .61391 | .55839 |
| 11 | .89632 | .87228 | .84893 | .80426 | .76214 | .72242 | .64958 | .58468 | .52679 |
| 12 | .88745 | .86151 | .83639 | .78849 | .74356 | .70138 | .62460 | .55684 | .49697 |
| 13 | .87866 | .85087 | .82403 | .77303 | .72542 | .68095 | .60057 | .53032 | .46884 |
| 14 | .86996 | .84037 | .81185 | .75788 | .70773 | .66112 | .57748 | .50507 | .44230 |
| 15 | .86135 | .82999 | .79985 | .74301 | .69047 | .64186 | .55526 | .48102 | .41727 |
| 16 | .85282 | .81975 | .78803 | .72845 | .67362 | .62317 | .53391 | .45811 | .39365 |
| 17 | .84438 | .80963 | .77639 | .71416 | .65720 | .60502 | .51337 | .43630 | .37136 |
| 18 | .83602 | .79963 | .76491 | .70016 | .64117 | .58739 | .49363 | .41552 | .35034 |
| 19 | .82774 | .78976 | .75361 | .68643 | .62553 | .57029 | .47464 | .39573 | .33051 |
| 20 | .81954 | .78001 | .74247 | .67297 | .61027 | .55368 | .45639 | .37689 | .31180 |
| 21 | .81143 | .77038 | .73150 | .65978 | .59539 | .53755 | .43883 | .35894 | .29416 |
| 22 | .80340 | .76087 | .72069 | .64684 | .58086 | .52189 | .42196 | .34185 | .27751 |
| 23 | .79544 | .75147 | .71004 | .63416 | .56670 | .50669 | .40573 | .32557 | .26180 |
| 24 | .78757 | .74220 | .69954 | .62172 | .55288 | .49193 | .39012 | .31007 | .24698 |
| 25 | .77977 | .73303 | .68921 | .60953 | .53939 | .47761 | .37512 | .29530 | .23300 |
| 26 | .77205 | .72398 | .67902 | .59758 | .52623 | .46369 | .36069 | .28124 | .21981 |
| 27 | .76440 | .71505 | .66899 | .58586 | .51340 | .45019 | .34682 | .26785 | .20737 |
| 28 | .75684 | .70622 | .65910 | .57437 | .50088 | .43708 | .33348 | .25509 | .19563 |
| 29 | .74934 | .69750 | .64936 | .56311 | .48866 | .42435 | .32065 | .24295 | .18456 |
| 30 | .74192 | .68889 | .63976 | .55207 | .47674 | .41199 | .30832 | .23138 | .17411 |
| 31 | .73458 | .68038 | .63031 | .54125 | .46511 | .39999 | .29646 | .22036 | .16425 |
| 32 | .72730 | .67198 | .62099 | .53063 | .45377 | .38834 | .28506 | .20987 | .15496 |
| 33 | .72010 | .66369 | .61182 | .52023 | .44270 | .37703 | .27409 | .19987 | .14619 |
| 34 | .71297 | .65549 | .60277 | .51003 | .43191 | .36604 | .26355 | .19035 | .13791 |
| 35 | .70591 | .64740 | .59387 | .50003 | .42137 | .35538 | .25342 | .18129 | .13011 |
| 36 | .69892 | .63941 | .58509 | .49022 | .41109 | .34503 | .24367 | .17266 | .12274 |
| 37 | .69200 | .63152 | .57644 | .48061 | .40107 | .33498 | .23430 | .16444 | .11579 |
| 38 | .68515 | .62372 | .56792 | .47119 | .39128 | .32523 | .22529 | .15661 | .10924 |
| 39 | .67837 | .61602 | .55953 | .46195 | .38174 | .31575 | .21662 | .14915 | .10306 |
| 40 | .67165 | .60841 | .55126 | .45289 | .37243 | .30656 | .20829 | .14205 | .09722 |
| 41 | .66500 | .60090 | .54312 | .44401 | .36335 | .29763 | .20028 | .13528 | .09172 |
| 42 | .65842 | .59348 | .53509 | .43530 | .35448 | .28896 | .19257 | .12884 | .08653 |
| 43 | .65190 | .58616 | .52718 | .42677 | .34584 | .28054 | .18517 | .12270 | .08163 |
| 44 | .64545 | .57892 | .51939 | .41840 | .33740 | .27237 | .17805 | .11686 | .07701 |
| 45 | .63905 | .57177 | .51171 | .41020 | .32917 | .26444 | .17120 | .11130 | .07265 |
| 46 | .63273 | .56471 | .50415 | .40215 | .32115 | .25674 | .16461 | .10600 | .06854 |
| 47 | .62646 | .55774 | .49670 | .39427 | .31331 | .24926 | .15828 | .10095 | .06466 |
| 48 | .62026 | .55086 | .48936 | .38654 | .30567 | .24200 | .15219 | .09614 | .06100 |
| 49 | .61412 | .54406 | .48213 | .37896 | .29822 | .23495 | .14634 | .09156 | .05755 |
| 50 | .60804 | .53734 | .47500 | .37153 | .29094 | .22811 | .14071 | .08720 | .05429 |

## TABLE

## 25

## AMOUNT OF AN ANNUITY: $\dfrac{(1+r)^n - 1}{r}$

If a principal $P$ is deposited at the end of each year at interest rate $r$ (in decimals) compounded annually, then at the end of $n$ years the accumulated amount is $P\left[\dfrac{(1+r)^n - 1}{r}\right]$. The process is often called an *annuity*.

| $n$ \ $r$ | 1% | $1\frac{1}{4}$% | $1\frac{1}{2}$% | 2% | $2\frac{1}{2}$% | 3% | 4% | 5% | 6% |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 2 | 2.0100 | 2.0125 | 2.0150 | 2.0200 | 2.0250 | 2.0300 | 2.0400 | 2.0500 | 2.0600 |
| 3 | 3.0301 | 3.0377 | 3.0452 | 3.0604 | 3.0756 | 3.0909 | 3.1216 | 3.1525 | 3.1836 |
| 4 | 4.0604 | 4.0756 | 4.0909 | 4.1216 | 4.1525 | 4.1836 | 4.2465 | 4.3101 | 4.3746 |
| 5 | 5.1010 | 5.1266 | 5.1523 | 5.2040 | 5.2563 | 5.3091 | 5.4163 | 5.5256 | 5.6371 |
| 6 | 6.1520 | 6.1907 | 6.2296 | 6.3081 | 6.3877 | 6.4684 | 6.6330 | 6.8019 | 6.9753 |
| 7 | 7.2135 | 7.2680 | 7.3230 | 7.4343 | 7.5474 | 7.6625 | 7.8983 | 8.1420 | 8.3938 |
| 8 | 8.2857 | 8.3589 | 8.4328 | 8.5830 | 8.7361 | 8.8923 | 9.2142 | 9.5491 | 9.8975 |
| 9 | 9.3685 | 9.4634 | 9.5593 | 9.7546 | 9.9545 | 10.1591 | 10.5828 | 11.0266 | 11.4913 |
| 10 | 10.4622 | 10.5817 | 10.7027 | 10.9497 | 11.2034 | 11.4639 | 12.0061 | 12.5779 | 13.1808 |
| 11 | 11.5668 | 11.7139 | 11.8633 | 12.1687 | 12.4835 | 12.8078 | 13.4864 | 14.2068 | 14.9716 |
| 12 | 12.6825 | 12.8604 | 13.0412 | 13.4121 | 13.7956 | 14.1920 | 15.0258 | 15.9171 | 16.8699 |
| 13 | 13.8093 | 14.0211 | 14.2368 | 14.6803 | 15.1404 | 15.6178 | 16.6268 | 17.7130 | 18.8821 |
| 14 | 14.9474 | 15.1964 | 15.4504 | 15.9739 | 16.5190 | 17.0863 | 18.2919 | 19.5986 | 21.0151 |
| 15 | 16.0969 | 16.3863 | 16.6821 | 17.2934 | 17.9319 | 18.5989 | 20.0236 | 21.5786 | 23.2760 |
| 16 | 17.2579 | 17.5912 | 17.9324 | 18.6393 | 19.3802 | 20.1569 | 21.8245 | 23.6575 | 25.6725 |
| 17 | 18.4304 | 18.8111 | 19.2014 | 20.0121 | 20.8647 | 21.7616 | 23.6975 | 25.8404 | 28.2129 |
| 18 | 19.6147 | 20.0462 | 20.4894 | 21.4123 | 22.3863 | 23.4144 | 25.6454 | 28.1324 | 30.9057 |
| 19 | 20.8109 | 21.2968 | 21.7967 | 22.8406 | 23.9460 | 25.1169 | 27.6712 | 30.5390 | 33.7600 |
| 20 | 22.0190 | 22.5630 | 23.1237 | 24.2974 | 25.5447 | 26.8704 | 29.7781 | 33.0660 | 36.7856 |
| 21 | 23.2392 | 23.8450 | 24.4705 | 25.7833 | 27.1833 | 28.6765 | 31.9692 | 35.7193 | 39.9927 |
| 22 | 24.4716 | 25.1431 | 25.8376 | 27.2990 | 28.8629 | 30.5368 | 34.2480 | 38.5052 | 43.3923 |
| 23 | 25.7163 | 26.4574 | 27.2251 | 28.8450 | 30.5844 | 32.4529 | 36.6179 | 41.4305 | 46.9958 |
| 24 | 26.9735 | 27.7881 | 28.6335 | 30.4219 | 32.3490 | 34.4265 | 39.0826 | 44.5020 | 50.8156 |
| 25 | 28.2432 | 29.1354 | 30.0630 | 32.0303 | 34.1578 | 36.4593 | 41.6459 | 47.7271 | 54.8645 |
| 26 | 29.5256 | 30.4996 | 31.5140 | 33.6709 | 36.0117 | 38.5530 | 44.3117 | 51.1135 | 59.1564 |
| 27 | 30.8209 | 31.8809 | 32.9867 | 35.3443 | 37.9120 | 40.7096 | 47.0842 | 54.6691 | 63.7058 |
| 28 | 32.1291 | 33.2794 | 34.4815 | 37.0512 | 39.8598 | 42.9309 | 49.9676 | 58.4026 | 68.5281 |
| 29 | 33.4504 | 34.6954 | 35.9987 | 38.7922 | 41.8563 | 45.2189 | 52.9663 | 62.3227 | 73.6398 |
| 30 | 34.7849 | 36.1291 | 37.5387 | 40.5681 | 43.9027 | 47.5754 | 56.0849 | 66.4388 | 79.0582 |
| 31 | 36.1327 | 37.5807 | 39.1018 | 42.3794 | 46.0003 | 50.0027 | 59.3283 | 70.7608 | 84.8017 |
| 32 | 37.4941 | 39.0504 | 40.6883 | 44.2270 | 48.1503 | 52.5028 | 62.7015 | 75.2988 | 90.8898 |
| 33 | 38.8690 | 40.5386 | 42.2986 | 46.1116 | 50.3540 | 55.0778 | 66.2095 | 80.0638 | 97.3432 |
| 34 | 40.2577 | 42.0453 | 43.9331 | 48.0338 | 52.6129 | 57.7302 | 69.8579 | 85.0670 | 104.1838 |
| 35 | 41.6603 | 43.5709 | 45.5921 | 49.9945 | 54.9282 | 60.4621 | 73.6522 | 90.3203 | 111.4348 |
| 36 | 43.0769 | 45.1155 | 47.2760 | 51.9944 | 57.3014 | 63.2759 | 77.5983 | 95.8363 | 119.1209 |
| 37 | 44.5076 | 46.6794 | 48.9851 | 54.0343 | 59.7339 | 66.1742 | 81.7022 | 101.6281 | 127.2681 |
| 38 | 45.9527 | 48.2629 | 50.7199 | 56.1149 | 62.2273 | 69.1594 | 85.9703 | 107.7095 | 135.9042 |
| 39 | 47.4123 | 49.8662 | 52.4807 | 58.2372 | 64.7830 | 72.2342 | 90.4091 | 114.0950 | 145.0585 |
| 40 | 48.8864 | 51.4896 | 54.2679 | 60.4020 | 67.4026 | 75.4013 | 95.0255 | 120.7998 | 154.7620 |
| 41 | 50.3752 | 53.1332 | 56.0819 | 62.6100 | 70.0876 | 78.6633 | 99.8265 | 127.8398 | 165.0477 |
| 42 | 51.8790 | 54.7973 | 57.9231 | 64.8622 | 72.8398 | 82.0232 | 104.8196 | 135.2318 | 175.9505 |
| 43 | 53.3978 | 56.4823 | 59.7920 | 67.1595 | 75.6608 | 85.4839 | 110.0124 | 142.9933 | 187.5076 |
| 44 | 54.9318 | 58.1883 | 61.6889 | 69.5027 | 78.5523 | 89.0484 | 115.4129 | 151.1430 | 199.7580 |
| 45 | 56.4811 | 59.9157 | 63.6142 | 71.8927 | 81.5161 | 92.7199 | 121.0294 | 159.7002 | 212.7435 |
| 46 | 58.0459 | 61.6646 | 65.5684 | 74.3306 | 84.5540 | 96.5015 | 126.8706 | 168.6852 | 226.5081 |
| 47 | 59.6263 | 63.4354 | 67.5519 | 76.8172 | 87.6679 | 100.3965 | 132.9454 | 178.1194 | 241.0986 |
| 48 | 61.2226 | 65.2284 | 69.5652 | 79.3535 | 90.8596 | 104.4084 | 139.2632 | 188.0254 | 256.5645 |
| 49 | 62.8348 | 67.0437 | 71.6087 | 81.9406 | 94.1311 | 108.5406 | 145.8337 | 198.4267 | 272.9584 |
| 50 | 64.4632 | 68.8818 | 73.6828 | 84.5794 | 97.4843 | 112.7969 | 152.6671 | 209.3480 | 290.3359 |

**TABLE**

**26**

## PRESENT VALUE OF AN ANNUITY: $\frac{1-(1+r)^{-n}}{r}$

An annuity in which the yearly payment at the end of each of $n$ years is $A$ at an interest rate $r$ (in decimals) compounded annually has present value $A\left[\frac{1-(1+r)^{-n}}{r}\right]$.

| $n$ \ $r$ | 1% | $1\frac{1}{4}$% | $1\frac{1}{2}$% | 2% | $2\frac{1}{2}$% | 3% | 4% | 5% | 6% |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.9901 | 0.9877 | 0.9852 | 0.9804 | 0.9756 | 0.9709 | 0.9615 | 0.9524 | 0.9434 |
| 2 | 1.9704 | 1.9631 | 1.9559 | 1.9416 | 1.9274 | 1.9135 | 1.8861 | 1.8594 | 1.8334 |
| 3 | 2.9410 | 2.9265 | 2.9122 | 2.8839 | 2.8560 | 2.8286 | 2.7751 | 2.7232 | 2.6730 |
| 4 | 3.9020 | 3.8781 | 3.8544 | 3.8077 | 3.7620 | 3.7171 | 3.6299 | 3.5460 | 3.4651 |
| 5 | 4.8534 | 4.8178 | 4.7826 | 4.7135 | 4.6458 | 4.5797 | 4.4518 | 4.3295 | 4.2124 |
| 6 | 5.7955 | 5.7460 | 5.6972 | 5.6014 | 5.5081 | 5.4172 | 5.2421 | 5.0757 | 4.9173 |
| 7 | 6.7282 | 6.6627 | 6.5982 | 6.4720 | 6.3494 | 6.2303 | 6.0021 | 5.7864 | 5.5824 |
| 8 | 7.6517 | 7.5681 | 7.4859 | 7.3255 | 7.1701 | 7.0197 | 6.7327 | 6.4632 | 6.2098 |
| 9 | 8.5660 | 8.4623 | 8.3605 | 8.1622 | 7.9709 | 7.7861 | 7.4353 | 7.1078 | 6.8017 |
| 10 | 9.4713 | 9.3455 | 9.2222 | 8.9826 | 8.7521 | 8.5302 | 8.1109 | 7.7217 | 7.3601 |
| 11 | 10.3676 | 10.2178 | 10.0711 | 9.7868 | 9.5142 | 9.2526 | 8.7605 | 8.3064 | 7.8869 |
| 12 | 11.2551 | 11.0793 | 10.9075 | 10.5753 | 10.2578 | 9.9540 | 9.3851 | 8.8633 | 8.3838 |
| 13 | 12.1337 | 11.9302 | 11.7315 | 11.3484 | 10.9832 | 10.6350 | 9.9856 | 9.3936 | 8.8527 |
| 14 | 13.0037 | 12.7706 | 12.5434 | 12.1062 | 11.6909 | 11.2961 | 10.5631 | 9.8986 | 9.2950 |
| 15 | 13.8651 | 13.6005 | 13.3432 | 12.8493 | 12.3814 | 11.9379 | 11.1184 | 10.3797 | 9.7122 |
| 16 | 14.7179 | 14.4203 | 14.1313 | 13.5777 | 13.0550 | 12.5611 | 11.6523 | 10.8378 | 10.1059 |
| 17 | 15.5623 | 15.2299 | 14.9076 | 14.2919 | 13.7122 | 13.1661 | 12.1657 | 11.2741 | 10.4773 |
| 18 | 16.3983 | 16.0295 | 15.6726 | 14.9920 | 14.3534 | 13.7535 | 12.6593 | 11.6896 | 10.8276 |
| 19 | 17.2260 | 16.8193 | 16.4262 | 15.6785 | 14.9789 | 14.3238 | 13.1339 | 12.0853 | 11.1581 |
| 20 | 18.0456 | 17.5993 | 17.1686 | 16.3514 | 15.5892 | 14.8775 | 13.5903 | 12.4622 | 11.4699 |
| 21 | 18.8570 | 18.3697 | 17.9001 | 17.0112 | 16.1845 | 15.4150 | 14.0292 | 12.8212 | 11.7641 |
| 22 | 19.6604 | 19.1306 | 18.6208 | 17.6580 | 16.7654 | 15.9369 | 14.4511 | 13.1630 | 12.0416 |
| 23 | 20.4558 | 19.8820 | 19.3309 | 18.2922 | 17.3321 | 16.4436 | 14.8568 | 13.4886 | 12.3034 |
| 24 | 21.2434 | 20.6242 | 20.0304 | 18.9139 | 17.8850 | 16.9355 | 15.2470 | 13.7986 | 12.5504 |
| 25 | 22.0232 | 21.3573 | 20.7196 | 19.5235 | 18.4244 | 17.4131 | 15.6221 | 14.0939 | 12.7834 |
| 26 | 22.7952 | 22.0813 | 21.3986 | 20.1210 | 18.9506 | 17.8768 | 15.9828 | 14.3752 | 13.0032 |
| 27 | 23.5596 | 22.7963 | 22.0676 | 20.7069 | 19.4640 | 18.3270 | 16.3296 | 14.6430 | 13.2105 |
| 28 | 24.3164 | 23.5025 | 22.7267 | 21.2813 | 19.9649 | 18.7641 | 16.6631 | 14.8981 | 13.4062 |
| 29 | 25.0658 | 24.2000 | 23.3761 | 21.8444 | 20.4535 | 19.1885 | 16.9837 | 15.1411 | 13.5907 |
| 30 | 25.8077 | 24.8889 | 24.0158 | 22.3965 | 20.9303 | 19.6004 | 17.2920 | 15.3725 | 13.7648 |
| 31 | 26.5423 | 25.5693 | 24.6461 | 22.9377 | 21.3954 | 20.0004 | 17.5885 | 15.5928 | 13.9291 |
| 32 | 27.2696 | 26.2413 | 25.2671 | 23.4683 | 21.8492 | 20.3888 | 17.8736 | 15.8027 | 14.0840 |
| 33 | 27.9897 | 26.9050 | 25.8790 | 23.9886 | 22.2919 | 20.7658 | 18.1476 | 16.0025 | 14.2302 |
| 34 | 28.7027 | 27.5605 | 26.4817 | 24.4986 | 22.7238 | 21.1318 | 18.4112 | 16.1929 | 14.3681 |
| 35 | 29.4086 | 28.2079 | 27.0756 | 24.9986 | 23.1452 | 21.4872 | 18.6646 | 16.3742 | 14.4982 |
| 36 | 30.1075 | 28.8473 | 27.6607 | 25.4888 | 23.5563 | 21.8323 | 18.9083 | 16.5469 | 14.6210 |
| 37 | 30.7995 | 29.4788 | 28.2371 | 25.9695 | 23.9573 | 22.1672 | 19.1426 | 16.7113 | 14.7368 |
| 38 | 31.4847 | 30.1025 | 28.8051 | 26.4406 | 24.3486 | 22.4925 | 19.3679 | 16.8679 | 14.8460 |
| 39 | 32.1630 | 30.7185 | 29.3646 | 26.9026 | 24.7303 | 22.8082 | 19.5845 | 17.0170 | 14.9491 |
| 40 | 32.8347 | 31.3269 | 29.9158 | 27.3555 | 25.1028 | 23.1148 | 19.7928 | 17.1591 | 15.0463 |
| 41 | 33.4997 | 31.9278 | 30.4590 | 27.7995 | 25.4661 | 23.4124 | 19.9931 | 17.2944 | 15.1380 |
| 42 | 34.1581 | 32.5213 | 30.9941 | 28.2348 | 25.8206 | 23.7014 | 20.1856 | 17.4232 | 15.2245 |
| 43 | 34.8100 | 33.1075 | 31.5212 | 28.6616 | 26.1664 | 23.9819 | 20.3708 | 17.5459 | 15.3062 |
| 44 | 35.4555 | 33.6864 | 32.0406 | 29.0800 | 26.5038 | 24.2543 | 20.5488 | 17.6628 | 15.3832 |
| 45 | 36.0945 | 34.2582 | 32.5523 | 29.4902 | 26.8330 | 24.5187 | 20.7200 | 17.7741 | 15.4558 |
| 46 | 36.7272 | 34.8229 | 33.0565 | 29.8923 | 27.1542 | 24.7754 | 20.8847 | 17.8801 | 15.5244 |
| 47 | 37.3537 | 35.3806 | 33.5532 | 30.2866 | 27.4675 | 25.0247 | 21.0429 | 17.9810 | 15.5890 |
| 48 | 37.9740 | 35.9315 | 34.0426 | 30.6731 | 27.7732 | 25.2667 | 21.1951 | 18.0772 | 15.6500 |
| 49 | 38.5881 | 36.4755 | 34.5247 | 31.0521 | 28.0714 | 25.5017 | 21.3415 | 18.1687 | 15.7076 |
| 50 | 39.1961 | 37.0129 | 34.9997 | 31.4236 | 28.3623 | 25.7298 | 21.4822 | 18.2559 | 15.7619 |

## TABLE 27

## BESSEL FUNCTIONS
## $J_0(x)$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0. | 1.0000 | .9975 | .9900 | .9776 | .9604 | .9385 | .9120 | .8812 | .8463 | .8075 |
| 1. | .7652 | .7196 | .6711 | .6201 | .5669 | .5118 | .4554 | .3980 | .3400 | .2818 |
| 2. | .2239 | .1666 | .1104 | .0555 | .0025 | −.0484 | −.0968 | −.1424 | −.1850 | −.2243 |
| 3. | −.2601 | −.2921 | −.3202 | −.3443 | −.3643 | −.3801 | −.3918 | −.3992 | −.4026 | −.4018 |
| 4. | −.3971 | −.3887 | −.3766 | −.3610 | −.3423 | −.3205 | −.2961 | −.2693 | −.2404 | −.2097 |
| 5. | −.1776 | −.1443 | −.1103 | −.0758 | −.0412 | −.0068 | .0270 | .0599 | .0917 | .1220 |
| 6. | .1506 | .1773 | .2017 | .2238 | .2433 | .2601 | .2740 | .2851 | .2931 | .2981 |
| 7. | .3001 | .2991 | .2951 | .2882 | .2786 | .2663 | .2516 | .2346 | .2154 | .1944 |
| 8. | .1717 | .1475 | .1222 | .0960 | .0692 | .0419 | .0146 | −.0125 | −.0392 | −.0653 |
| 9. | −.0903 | −.1142 | −.1367 | −.1577 | −.1768 | −.1939 | −.2090 | −.2218 | −.2323 | −.2403 |

## TABLE 28

## BESSEL FUNCTIONS
## $J_1(x)$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0. | .0000 | .0499 | .0995 | .1483 | .1960 | .2423 | .2867 | .3290 | .3688 | .4059 |
| 1. | .4401 | .4709 | .4983 | .5220 | .5419 | .5579 | .5699 | .5778 | .5815 | .5812 |
| 2. | .5767 | .5683 | .5560 | .5399 | .5202 | .4971 | .4708 | .4416 | .4097 | .3754 |
| 3. | .3391 | .3009 | .2613 | .2207 | .1792 | .1374 | .0955 | .0538 | .0128 | −.0272 |
| 4. | −.0660 | −.1033 | −.1386 | −.1719 | −.2028 | −.2311 | −.2566 | −.2791 | −.2985 | −.3147 |
| 5. | −.3276 | −.3371 | −.3432 | −.3460 | −.3453 | −.3414 | −.3343 | −.3241 | −.3110 | −.2951 |
| 6. | −.2767 | −.2559 | −.2329 | −.2081 | −.1816 | −.1538 | −.1250 | −.0953 | −.0652 | −.0349 |
| 7. | −.0047 | .0252 | .0543 | .0826 | .1096 | .1352 | .1592 | .1813 | .2014 | .2192 |
| 8. | .2346 | .2476 | .2580 | .2657 | .2708 | .2731 | .2728 | .2697 | .2641 | .2559 |
| 9. | .2453 | .2324 | .2174 | .2004 | .1816 | .1613 | .1395 | .1166 | .0928 | .0684 |

## TABLE 29

## BESSEL FUNCTIONS

### $Y_0(x)$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0. | $-\infty$ | −1.5342 | −1.0811 | −.8073 | −.6060 | −.4445 | −.3085 | −.1907 | −.0868 | .0056 |
| 1. | .0883 | .1622 | .2281 | .2865 | .3379 | .3824 | .4204 | .4520 | .4774 | .4968 |
| 2. | .5104 | .5183 | .5208 | .5181 | .5104 | .4981 | .4813 | .4605 | .4359 | .4079 |
| 3. | .3769 | .3431 | .3071 | .2691 | .2296 | .1890 | .1477 | .1061 | .0645 | .0234 |
| 4. | −.0169 | −.0561 | −.0938 | −.1296 | −.1633 | −.1947 | −.2235 | −.2494 | −.2723 | −.2921 |
| 5. | −.3085 | −.3216 | −.3313 | −.3374 | −.3402 | −.3395 | −.3354 | −.3282 | −.3177 | −.3044 |
| 6. | −.2882 | −.2694 | −.2483 | −.2251 | −.1999 | −.1732 | −.1452 | −.1162 | −.0864 | −.0563 |
| 7. | −.0259 | .0042 | .0339 | .0628 | .0907 | .1173 | .1424 | .1658 | .1872 | .2065 |
| 8. | .2235 | .2381 | .2501 | .2595 | .2662 | .2702 | .2715 | .2700 | .2659 | .2592 |
| 9. | .2499 | .2383 | .2245 | .2086 | .1907 | .1712 | .1502 | .1279 | .1045 | .0804 |

## TABLE 30

## BESSEL FUNCTIONS

### $Y_1(x)$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0. | $-\infty$ | −6.4590 | −3.3238 | −2.2931 | −1.7809 | −1.4715 | −1.2604 | −1.1032 | −.9781 | −.8731 |
| 1. | −.7812 | −.6981 | −.6211 | −.5485 | −.4791 | −.4123 | −.3476 | −.2847 | −.2237 | −.1644 |
| 2. | −.1070 | −.0517 | .0015 | .0523 | .1005 | .1459 | .1884 | .2276 | .2635 | .2959 |
| 3. | .3247 | .3496 | .3707 | .3879 | .4010 | .4102 | .4154 | .4167 | .4141 | .4078 |
| 4. | .3979 | .3846 | .3680 | .3484 | .3260 | .3010 | .2737 | .2445 | .2136 | .1812 |
| 5. | .1479 | .1137 | .0792 | .0445 | .0101 | −.0238 | −.0568 | −.0887 | −.1192 | −.1481 |
| 6. | −.1750 | −.1998 | −.2223 | −.2422 | −.2596 | −.2741 | −.2857 | −.2945 | −.3002 | −.3029 |
| 7. | −.3027 | −.2995 | −.2934 | −.2846 | −.2731 | −.2591 | −.2428 | −.2243 | −.2039 | −.1817 |
| 8. | −.1581 | −.1331 | −.1072 | −.0806 | −.0535 | −.0262 | .0011 | .0280 | .0544 | .0799 |
| 9. | .1043 | .1275 | .1491 | .1691 | .1871 | .2032 | .2171 | .2287 | .2379 | .2447 |

## TABLE 31

## BESSEL FUNCTIONS $I_0(x)$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0. | 1.000 | 1.003 | 1.010 | 1.023 | 1.040 | 1.063 | 1.092 | 1.126 | 1.167 | 1.213 |
| 1. | 1.266 | 1.326 | 1.394 | 1.469 | 1.553 | 1.647 | 1.750 | 1.864 | 1.990 | 2.128 |
| 2. | 2.280 | 2.446 | 2.629 | 2.830 | 3.049 | 3.290 | 3.553 | 3.842 | 4.157 | 4.503 |
| 3. | 4.881 | 5.294 | 5.747 | 6.243 | 6.785 | 7.378 | 8.028 | 8.739 | 9.517 | 10.37 |
| 4. | 11.30 | 12.32 | 13.44 | 14.67 | 16.01 | 17.48 | 19.09 | 20.86 | 22.79 | 24.91 |
| 5. | 27.24 | 29.79 | 32.58 | 35.65 | 39.01 | 42.69 | 46.74 | 51.17 | 56.04 | 61.38 |
| 6. | 67.23 | 73.66 | 80.72 | 88.46 | 96.96 | 106.3 | 116.5 | 127.8 | 140.1 | 153.7 |
| 7. | 168.6 | 185.0 | 202.9 | 222.7 | 244.3 | 268.2 | 294.3 | 323.1 | 354.7 | 389.4 |
| 8. | 427.6 | 469.5 | 515.6 | 566.3 | 621.9 | 683.2 | 750.5 | 824.4 | 905.8 | 995.2 |
| 9. | 1094 | 1202 | 1321 | 1451 | 1595 | 1753 | 1927 | 2119 | 2329 | 2561 |

## TABLE 32

## BESSEL FUNCTIONS $I_1(x)$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0. | .0000 | .0501 | .1005 | .1517 | .2040 | .2579 | .3137 | .3719 | .4329 | .4971 |
| 1. | .5652 | .6375 | .7147 | .7973 | .8861 | .9817 | 1.085 | 1.196 | 1.317 | 1.448 |
| 2. | 1.591 | 1.745 | 1.914 | 2.098 | 2.298 | 2.517 | 2.755 | 3.016 | 3.301 | 3.613 |
| 3. | 3.953 | 4.326 | 4.734 | 5.181 | 5.670 | 6.206 | 6.793 | 7.436 | 8.140 | 8.913 |
| 4. | 9.759 | 10.69 | 11.71 | 12.82 | 14.05 | 15.39 | 16.86 | 18.48 | 20.25 | 22.20 |
| 5. | 24.34 | 26.68 | 29.25 | 32.08 | 35.18 | 38.59 | 42.33 | 46.44 | 50.95 | 55.90 |
| 6. | 61.34 | 67.32 | 73.89 | 81.10 | 89.03 | 97.74 | 107.3 | 117.8 | 129.4 | 142.1 |
| 7. | 156.0 | 171.4 | 188.3 | 206.8 | 227.2 | 249.6 | 274.2 | 301.3 | 331.1 | 363.9 |
| 8. | 399.9 | 439.5 | 483.0 | 531.0 | 583.7 | 641.6 | 705.4 | 775.5 | 852.7 | 937.5 |
| 9. | 1031 | 1134 | 1247 | 1371 | 1508 | 1658 | 1824 | 2006 | 2207 | 2428 |

## TABLE 33

## BESSEL FUNCTIONS

## $K_0(x)$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0. | $\infty$ | 2.4271 | 1.7527 | 1.3725 | 1.1145 | .9244 | .7775 | .6605 | .5653 | .4867 |
| 1. | .4210 | .3656 | .3185 | .2782 | .2437 | .2138 | .1880 | .1655 | .1459 | .1288 |
| 2. | .1139 | .1008 | .08927 | .07914 | .07022 | .06235 | .05540 | .04926 | .04382 | .03901 |
| 3. | .03474 | .03095 | .02759 | .02461 | .02196 | .01960 | .01750 | .01563 | .01397 | .01248 |
| 4. | .01116 | $.0^{2}9980$ | $.0^{2}8927$ | $.0^{2}7988$ | $.0^{2}7149$ | $.0^{2}6400$ | $.0^{2}5730$ | $.0^{2}5132$ | $.0^{2}4597$ | $.0^{2}4119$ |
| 5. | $.0^{2}3691$ | $.0^{2}3308$ | $.0^{2}2966$ | $.0^{2}2659$ | $.0^{2}2385$ | $.0^{2}2139$ | $.0^{2}1918$ | $.0^{2}1721$ | $.0^{2}1544$ | $.0^{2}1386$ |
| 6. | $.0^{2}1244$ | $.0^{2}1117$ | $.0^{2}1003$ | $.0^{3}9001$ | $.0^{3}8083$ | $.0^{3}7259$ | $.0^{3}6520$ | $.0^{3}5857$ | $.0^{3}5262$ | $.0^{3}4728$ |
| 7. | $.0^{3}4248$ | $.0^{3}3817$ | $.0^{3}3431$ | $.0^{3}3084$ | $.0^{3}2772$ | $.0^{3}2492$ | $.0^{3}2240$ | $.0^{3}2014$ | $.0^{3}1811$ | $.0^{3}1629$ |
| 8. | $.0^{3}1465$ | $.0^{3}1317$ | $.0^{3}1185$ | $.0^{3}1066$ | $.0^{4}9588$ | $.0^{4}8626$ | $.0^{4}7761$ | $.0^{4}6983$ | $.0^{4}6283$ | $.0^{4}5654$ |
| 9. | $.0^{4}5088$ | $.0^{4}4579$ | $.0^{4}4121$ | $.0^{4}3710$ | $.0^{4}3339$ | $.0^{4}3006$ | $.0^{4}2706$ | $.0^{4}2436$ | $.0^{4}2193$ | $.0^{4}1975$ |

## TABLE 34

## BESSEL FUNCTIONS

## $K_1(x)$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0. | $\infty$ | 9.8538 | 4.7760 | 3.0560 | 2.1844 | 1.6564 | 1.3028 | 1.0503 | .8618 | .7165 |
| 1. | .6019 | .5098 | .4346 | .3725 | .3208 | .2774 | .2406 | .2094 | .1826 | .1597 |
| 2. | .1399 | .1227 | .1079 | .09498 | .08372 | .07389 | .06528 | .05774 | .05111 | .04529 |
| 3. | .04016 | .03563 | .03164 | .02812 | .02500 | .02224 | .01979 | .01763 | .01571 | .01400 |
| 4. | .01248 | .01114 | $.0^{2}9938$ | $.0^{2}8872$ | $.0^{2}7923$ | $.0^{2}7078$ | $.0^{2}6325$ | $.0^{2}5654$ | $.0^{2}5055$ | $.0^{2}4521$ |
| 5. | $.0^{2}4045$ | $.0^{2}3619$ | $.0^{2}3239$ | $.0^{2}2900$ | $.0^{2}2597$ | $.0^{2}2326$ | $.0^{2}2083$ | $.0^{2}1866$ | $.0^{2}1673$ | $.0^{2}1499$ |
| 6. | $.0^{2}1344$ | $.0^{2}1205$ | $.0^{2}1081$ | $.0^{3}9691$ | $.0^{3}8693$ | $.0^{3}7799$ | $.0^{3}6998$ | $.0^{3}6280$ | $.0^{3}5636$ | $.0^{3}5059$ |
| 7. | $.0^{3}4542$ | $.0^{3}4078$ | $.0^{3}3662$ | $.0^{3}3288$ | $.0^{3}2953$ | $.0^{3}2653$ | $.0^{3}2383$ | $.0^{3}2141$ | $.0^{3}1924$ | $.0^{3}1729$ |
| 8. | $.0^{3}1554$ | $.0^{3}1396$ | $.0^{3}1255$ | $.0^{3}1128$ | $.0^{3}1014$ | $.0^{4}9120$ | $.0^{4}8200$ | $.0^{4}7374$ | $.0^{4}6631$ | $.0^{4}5964$ |
| 9. | $.0^{4}5364$ | $.0^{4}4825$ | $.0^{4}4340$ | $.0^{4}3904$ | $.0^{4}3512$ | $.0^{4}3160$ | $.0^{4}2843$ | $.0^{4}2559$ | $.0^{4}2302$ | $.0^{4}2072$ |

## TABLE 35

## BESSEL FUNCTIONS

## **Ber ($x$)**

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0. | 1.0000 | 1.0000 | 1.0000 | .9999 | .9996 | .9990 | .9980 | .9962 | .9936 | .9898 |
| 1. | .9844 | .9771 | .9676 | .9554 | .9401 | .9211 | .8979 | .8700 | .8367 | .7975 |
| 2. | .7517 | .6987 | .6377 | .5680 | .4890 | .4000 | .3001 | .1887 | .06511 | −.07137 |
| 3. | −.2214 | −.3855 | −.5644 | −.7584 | −.9680 | −1.1936 | −1.4353 | −1.6933 | −1.9674 | −2.2576 |
| 4. | −2.5634 | −2.8843 | −3.2195 | −3.5679 | −3.9283 | −4.2991 | −4.6784 | −5.0639 | −5.4531 | −5.8429 |
| 5. | −6.2301 | −6.6107 | −6.9803 | −7.3344 | −7.6674 | −7.9736 | −8.2466 | −8.4794 | −8.6644 | −8.7937 |
| 6. | −8.8583 | −8.8491 | −8.7561 | −8.5688 | −8.2762 | −7.8669 | −7.3287 | −6.6492 | −5.8155 | −4.8146 |
| 7. | −3.6329 | −2.2571 | −.6737 | 1.1308 | 3.1695 | 5.4550 | 7.9994 | 10.814 | 13.909 | 17.293 |
| 8. | 20.974 | 24.957 | 29.245 | 33.840 | 38.738 | 43.936 | 49.423 | 55.187 | 61.210 | 67.469 |
| 9. | 73.936 | 80.576 | 87.350 | 94.208 | 101.10 | 107.95 | 114.70 | 121.26 | 127.54 | 133.43 |

## TABLE 36

## BESSEL FUNCTIONS

## **Bei ($x$)**

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0. | .0000 | .0$^{2}$2500 | .01000 | .02250 | .04000 | .06249 | .08998 | .1224 | .1599 | .2023 |
| 1. | .2496 | .3017 | .3587 | .4204 | .4867 | .5576 | .6327 | .7120 | .7953 | .8821 |
| 2. | .9723 | 1.0654 | 1.1610 | 1.2585 | 1.3575 | 1.4572 | 1.5569 | 1.6557 | 1.7529 | 1.8472 |
| 3. | 1.9376 | 2.0228 | 2.1016 | 2.1723 | 2.2334 | 2.2832 | 2.3199 | 2.3413 | 2.3454 | 2.3300 |
| 4. | 2.2927 | 2.2309 | 2.1422 | 2.0236 | 1.8726 | 1.6860 | 1.4610 | 1.1946 | .8837 | .5251 |
| 5. | .1160 | −.3467 | −.8658 | −1.4443 | −2.0845 | −2.7890 | −3.5597 | −4.3986 | −5.3068 | −6.2854 |
| 6. | −7.3347 | −8.4545 | −9.6437 | −10.901 | −12.223 | −13.607 | −15.047 | −16.538 | −18.074 | −19.644 |
| 7. | −21.239 | −22.848 | −24.456 | −26.049 | −27.609 | −29.116 | −30.548 | −31.882 | −33.092 | −34.147 |
| 8. | −35.017 | −35.667 | −36.061 | −36.159 | −35.920 | −35.298 | −34.246 | −32.714 | −30.651 | −28.003 |
| 9. | −24.713 | −20.724 | −15.976 | −10.412 | −3.9693 | 3.4106 | 11.787 | 21.218 | 31.758 | 43.459 |

## TABLE 37

## BESSEL FUNCTIONS

## Ker ($x$)

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0. | $\infty$ | 2.4205 | 1.7331 | 1.3372 | 1.0626 | .8559 | .6931 | .5614 | .4529 | .3625 |
| 1. | .2867 | .2228 | .1689 | .1235 | .08513 | .05293 | .02603 | $.0^2 3691$ | −.01470 | −.02966 |
| 2. | −.04166 | −.05111 | −.05834 | −.06367 | −.06737 | −.06969 | −.07083 | −.07097 | −.07030 | −.06894 |
| 3. | −.06703 | −.06468 | −.06198 | −.05903 | −.05590 | −.05264 | −.04932 | −.04597 | −.04265 | −.03937 |
| 4. | −.03618 | −.03308 | −.03011 | −.02726 | −.02456 | −.02200 | −.01960 | −.01734 | −.01525 | −.01330 |
| 5. | −.01151 | $-.0^2 9865$ | $-.0^2 8359$ | $-.0^2 6989$ | $-.0^2 5749$ | $-.0^2 4632$ | $-.0^2 3632$ | $-.0^2 2740$ | $-.0^2 1952$ | $-.0^2 1258$ |
| 6. | $-.0^3 6530$ | $-.0^3 1295$ | $.0^3 3191$ | $.0^3 6991$ | $.0^2 1017$ | $.0^2 1278$ | $.0^2 1488$ | $.0^2 1653$ | $.0^2 1777$ | $.0^2 1866$ |
| 7. | $.0^2 1922$ | $.0^2 1951$ | $.0^2 1956$ | $.0^2 1940$ | $.0^2 1907$ | $.0^2 1860$ | $.0^2 1800$ | $.0^2 1731$ | $.0^2 1655$ | $.0^2 1572$ |
| 8. | $.0^2 1486$ | $.0^2 1397$ | $.0^2 1306$ | $.0^2 1216$ | $.0^2 1126$ | $.0^2 1037$ | $.0^3 9511$ | $.0^3 8675$ | $.0^3 7871$ | $.0^3 7102$ |
| 9. | $.0^3 6372$ | $.0^3 5681$ | $.0^3 5030$ | $.0^3 4422$ | $.0^3 3855$ | $.0^3 3330$ | $.0^3 2846$ | $.0^3 2402$ | $.0^3 1996$ | $.0^3 1628$ |

## TABLE 38

## BESSEL FUNCTIONS

## Kei ($x$)

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0. | −.7854 | −.7769 | −.7581 | −.7331 | −.7038 | −.6716 | −.6374 | −.6022 | −.5664 | −.5305 |
| 1. | −.4950 | −.4601 | −.4262 | −.3933 | −.3617 | −.3314 | −.3026 | −.2752 | −.2494 | −.2251 |
| 2. | −.2024 | −.1812 | −.1614 | −.1431 | −.1262 | −.1107 | −.09644 | −.08342 | −.07157 | −.06083 |
| 3. | −.05112 | −.04240 | −.03458 | −.02762 | −.02145 | −.01600 | −.01123 | $-.0^2 7077$ | $-.0^2 3487$ | $-.0^3 4108$ |
| 4. | $.0^2 2198$ | $.0^2 4386$ | $.0^2 6194$ | $.0^2 7661$ | $.0^2 8826$ | $.0^2 9721$ | .01038 | .01083 | .01110 | .01121 |
| 5. | .01119 | .01105 | .01082 | .01051 | .01014 | $.0^2 9716$ | $.0^2 9255$ | $.0^2 8766$ | $.0^2 8258$ | $.0^2 7739$ |
| 6. | $.0^2 7216$ | $.0^2 6696$ | $.0^2 6183$ | $.0^2 5681$ | $.0^2 5194$ | $.0^2 4724$ | $.0^2 4274$ | $.0^2 3846$ | $.0^2 3440$ | $.0^2 3058$ |
| 7. | $.0^2 2700$ | $.0^2 2366$ | $.0^2 2057$ | $.0^2 1770$ | $.0^2 1507$ | $.0^2 1267$ | $.0^2 1048$ | $.0^3 8498$ | $.0^3 6714$ | $.0^3 5117$ |
| 8. | $.0^3 3696$ | $.0^3 2440$ | $.0^3 1339$ | $.0^4 3809$ | $-.0^4 4449$ | $-.0^3 1149$ | $-.0^3 1742$ | $-.0^3 2233$ | $-.0^3 2632$ | $-.0^3 2949$ |
| 9. | $-.0^3 3192$ | $-.0^3 3368$ | $-.0^3 3486$ | $-.0^3 3552$ | $-.0^3 3574$ | $-.0^3 3557$ | $-.0^3 3508$ | $-.0^3 3430$ | $-.0^3 3329$ | $-.0^3 3210$ |

## TABLE
## 39

## VALUES FOR APPROXIMATE ZEROS OF BESSEL FUNCTIONS

The following table lists the first few positive roots of various equations. Note that for all cases listed the successive large roots differ approximately by $\pi = 3.14159\ldots$

| | $n = 0$ | $n = 1$ | $n = 2$ | $n = 3$ | $n = 4$ | $n = 5$ | $n = 6$ |
|---|---|---|---|---|---|---|---|
| $J_n(x) = 0$ | 2.4048 | 3.8317 | 5.1356 | 6.3802 | 7.5883 | 8.7715 | 9.9361 |
| | 5.5201 | 7.0156 | 8.4172 | 9.7610 | 11.0647 | 12.3386 | 13.5893 |
| | 8.6537 | 10.1735 | 11.6198 | 13.0152 | 14.3725 | 15.7002 | 17.0038 |
| | 11.7915 | 13.3237 | 14.7960 | 16.2235 | 17.6160 | 18.9801 | 20.3208 |
| | 14.9309 | 16.4706 | 17.9598 | 19.4094 | 20.8269 | 22.2178 | 23.5861 |
| | 18.0711 | 19.6159 | 21.1170 | 22.5827 | 24.0190 | 25.4303 | 26.8202 |
| $Y_n(x) = 0$ | 0.8936 | 2.1971 | 3.3842 | 4.5270 | 5.6452 | 6.7472 | 7.8377 |
| | 3.9577 | 5.4297 | 6.7938 | 8.0976 | 9.3616 | 10.5972 | 11.8110 |
| | 7.0861 | 8.5960 | 10.0235 | 11.3965 | 12.7301 | 14.0338 | 15.3136 |
| | 10.2223 | 11.7492 | 13.2100 | 14.6231 | 15.9996 | 17.3471 | 18.6707 |
| | 13.3611 | 14.8974 | 16.3790 | 17.8185 | 19.2244 | 20.6029 | 21.9583 |
| | 16.5009 | 18.0434 | 19.5390 | 20.9973 | 22.4248 | 23.8265 | 25.2062 |
| $J'_n(x) = 0$ | 0.0000 | 1.8412 | 3.0542 | 4.2012 | 5.3176 | 6.4156 | 7.5013 |
| | 3.8317 | 5.3314 | 6.7061 | 8.0152 | 9.2824 | 10.5199 | 11.7349 |
| | 7.0156 | 8.5363 | 9.9695 | 11.3459 | 12.6819 | 13.9872 | 15.2682 |
| | 10.1735 | 11.7060 | 13.1704 | 14.5859 | 15.9641 | 17.3128 | 18.6374 |
| | 13.3237 | 14.8636 | 16.3475 | 17.7888 | 19.1960 | 20.5755 | 21.9317 |
| | 16.4706 | 18.0155 | 19.5129 | 20.9725 | 22.4010 | 23.8036 | 25.1839 |
| $Y'_n(x) = 0$ | 2.1971 | 3.6830 | 5.0026 | 6.2536 | 7.4649 | 8.6496 | 9.8148 |
| | 5.4297 | 6.9415 | 8.3507 | 9.6988 | 11.0052 | 12.2809 | 13.5328 |
| | 8.5960 | 10.1234 | 11.5742 | 12.9724 | 14.3317 | 15.6608 | 16.9655 |
| | 11.7492 | 13.2858 | 14.7609 | 16.1905 | 17.5844 | 18.9497 | 20.2913 |
| | 14.8974 | 16.4401 | 17.9313 | 19.3824 | 20.8011 | 22.1928 | 23.5619 |
| | 18.0434 | 19.5902 | 21.0929 | 22.5598 | 23.9970 | 25.4091 | 26.7995 |

**TABLE**

**40**

## EXPONENTIAL, SINE AND COSINE INTEGRALS

$$Ei(x) = \int_x^\infty \frac{e^{-u}}{u}\,du, \quad Si(x) = \int_0^x \frac{\sin u}{u}\,du, \quad Ci(x) = \int_x^\infty \frac{\cos u}{u}\,du$$

| $x$ | $Ei(x)$ | $Si(x)$ | $Ci(x)$ |
|---|---|---|---|
| .0 | $\infty$ | .0000 | $\infty$ |
| .5 | .5598 | .4931 | .1778 |
| 1.0 | .2194 | .9461 | −.3374 |
| 1.5 | .1000 | 1.3247 | −.4704 |
| 2.0 | .04890 | 1.6054 | −.4230 |
| 2.5 | .02491 | 1.7785 | −.2859 |
| 3.0 | .01305 | 1.8487 | −.1196 |
| 3.5 | $.0^2 6970$ | 1.8331 | .0321 |
| 4.0 | $.0^2 3779$ | 1.7582 | .1410 |
| 4.5 | $.0^2 2073$ | 1.6541 | .1935 |
| 5.0 | $.0^2 1148$ | 1.5499 | .1900 |
| 5.5 | $.0^3 6409$ | 1.4687 | .1421 |
| 6.0 | $.0^3 3601$ | 1.4247 | .0681 |
| 6.5 | $.0^3 2034$ | 1.4218 | −.0111 |
| 7.0 | $.0^3 1155$ | 1.4546 | −.0767 |
| 7.5 | $.0^4 6583$ | 1.5107 | −.1156 |
| 8.0 | $.0^4 3767$ | 1.5742 | −.1224 |
| 8.5 | $.0^4 2162$ | 1.6296 | −.09943 |
| 9.0 | $.0^4 1245$ | 1.6650 | −.05535 |
| 9.5 | $.0^5 7185$ | 1.6745 | $-.0^2 2678$ |
| 10.0 | $.0^5 4157$ | 1.6583 | .04546 |

**TABLE**
**41**

## LEGENDRE POLYNOMIALS $P_n(x)$

$[P_0(x) = 1,\ P_1(x) = x]$

| $x$ | $P_2(x)$ | $P_3(x)$ | $P_4(x)$ | $P_5(x)$ |
|---|---|---|---|---|
| .00 | −.5000 | .0000 | .3750 | .0000 |
| .05 | −.4963 | −.0747 | .3657 | .0927 |
| .10 | −.4850 | −.1475 | .3379 | .1788 |
| .15 | −.4663 | −.2166 | .2928 | .2523 |
| .20 | −.4400 | −.2800 | .2320 | .3075 |
| .25 | −.4063 | −.3359 | .1577 | .3397 |
| .30 | −.3650 | −.3825 | .0729 | .3454 |
| .35 | −.3163 | −.4178 | −.0187 | .3225 |
| .40 | −.2600 | −.4400 | −.1130 | .2706 |
| .45 | −.1963 | −.4472 | −.2050 | .1917 |
| .50 | −.1250 | −.4375 | −.2891 | .0898 |
| .55 | −.0463 | −.4091 | −.3590 | −.0282 |
| .60 | .0400 | −.3600 | −.4080 | −.1526 |
| .65 | .1338 | −.2884 | −.4284 | −.2705 |
| .70 | .2350 | −.1925 | −.4121 | −.3652 |
| .75 | .3438 | −.0703 | −.3501 | −.4164 |
| .80 | .4600 | .0800 | −.2330 | −.3995 |
| .85 | .5838 | .2603 | −.0506 | −.2857 |
| .90 | .7150 | .4725 | .2079 | −.0411 |
| .95 | .8538 | .7184 | .5541 | .3727 |
| 1.00 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

TABLE

# 42

## LEGENDRE POLYNOMIALS $P_n(\cos\theta)$

$[P_0(\cos\theta) = 1]$

| $\theta$ | $P_1(\cos\theta)$ | $P_2(\cos\theta)$ | $P_3(\cos\theta)$ | $P_4(\cos\theta)$ | $P_5(\cos\theta)$ |
|---|---|---|---|---|---|
| 0° | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 5° | .9962 | .9886 | .9773 | .9623 | .9437 |
| 10° | .9848 | .9548 | .9106 | .8532 | .7840 |
| 15° | .9659 | .8995 | .8042 | .6847 | .5471 |
| 20° | .9397 | .8245 | .6649 | .4750 | .2715 |
| 25° | .9063 | .7321 | .5016 | .2465 | .0009 |
| 30° | .8660 | .6250 | .3248 | .0234 | −.2233 |
| 35° | .8192 | .5065 | .1454 | −.1714 | −.3691 |
| 40° | .7660 | .3802 | −.0252 | −.3190 | −.4197 |
| 45° | .7071 | .2500 | −.1768 | −.4063 | −.3757 |
| 50° | .6428 | .1198 | −.3002 | −.4275 | −.2545 |
| 55° | .5736 | −.0065 | −.3886 | −.3852 | −.0868 |
| 60° | .5000 | −.1250 | −.4375 | −.2891 | .0898 |
| 65° | .4226 | −.2321 | −.4452 | −.1552 | .2381 |
| 70° | .3420 | −.3245 | −.4130 | −.0038 | .3281 |
| 75° | .2588 | −.3995 | −.3449 | .1434 | .3427 |
| 80° | .1737 | −.4548 | −.2474 | .2659 | .2810 |
| 85° | .0872 | −.4886 | −.1291 | .3468 | .1577 |
| 90° | .0000 | −.5000 | .0000 | .3750 | .0000 |

## TABLE 43

## COMPLETE ELLIPTIC INTEGRALS OF FIRST AND SECOND KINDS

$$K = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}, \quad E = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta, \quad k = \sin \psi$$

| $\psi$ | $K$ | $E$ | $\psi$ | $K$ | $E$ | $\psi$ | $K$ | $E$ |
|---|---|---|---|---|---|---|---|---|
| 0° | 1.5708 | 1.5708 | 30° | 1.6858 | 1.4675 | 60° | 2.1565 | 1.2111 |
| 1 | 1.5709 | 1.5707 | 31 | 1.6941 | 1.4608 | 61 | 2.1842 | 1.2015 |
| 2 | 1.5713 | 1.5703 | 32 | 1.7028 | 1.4539 | 62 | 2.2132 | 1.1920 |
| 3 | 1.5719 | 1.5697 | 33 | 1.7119 | 1.4469 | 63 | 2.2435 | 1.1826 |
| 4 | 1.5727 | 1.5689 | 34 | 1.7214 | 1.4397 | 64 | 2.2754 | 1.1732 |
| 5 | 1.5738 | 1.5678 | 35 | 1.7312 | 1.4323 | 65 | 2.3088 | 1.1638 |
| 6 | 1.5751 | 1.5665 | 36 | 1.7415 | 1.4248 | 66 | 2.3439 | 1.1545 |
| 7 | 1.5767 | 1.5649 | 37 | 1.7522 | 1.4171 | 67 | 2.3809 | 1.1453 |
| 8 | 1.5785 | 1.5632 | 38 | 1.7633 | 1.4092 | 68 | 2.4198 | 1.1362 |
| 9 | 1.5805 | 1.5611 | 39 | 1.7748 | 1.4013 | 69 | 2.4610 | 1.1272 |
| 10 | 1.5828 | 1.5589 | 40 | 1.7868 | 1.3931 | 70 | 2.5046 | 1.1184 |
| 11 | 1.5854 | 1.5564 | 41 | 1.7992 | 1.3849 | 71 | 2.5507 | 1.1096 |
| 12 | 1.5882 | 1.5537 | 42 | 1.8122 | 1.3765 | 72 | 2.5998 | 1.1011 |
| 13 | 1.5913 | 1.5507 | 43 | 1.8256 | 1.3680 | 73 | 2.6521 | 1.0927 |
| 14 | 1.5946 | 1.5476 | 44 | 1.8396 | 1.3594 | 74 | 2.7081 | 1.0844 |
| 15 | 1.5981 | 1.5442 | 45 | 1.8541 | 1.3506 | 75 | 2.7681 | 1.0764 |
| 16 | 1.6020 | 1.5405 | 46 | 1.8691 | 1.3418 | 76 | 2.8327 | 1.0686 |
| 17 | 1.6061 | 1.5367 | 47 | 1.8848 | 1.3329 | 77 | 2.9026 | 1.0611 |
| 18 | 1.6105 | 1.5326 | 48 | 1.9011 | 1.3238 | 78 | 2.9786 | 1.0538 |
| 19 | 1.6151 | 1.5283 | 49 | 1.9180 | 1.3147 | 79 | 3.0617 | 1.0468 |
| 20 | 1.6200 | 1.5238 | 50 | 1.9356 | 1.3055 | 80 | 3.1534 | 1.0401 |
| 21 | 1.6252 | 1.5191 | 51 | 1.9539 | 1.2963 | 81 | 3.2553 | 1.0338 |
| 22 | 1.6307 | 1.5141 | 52 | 1.9729 | 1.2870 | 82 | 3.3699 | 1.0278 |
| 23 | 1.6365 | 1.5090 | 53 | 1.9927 | 1.2776 | 83 | 3.5004 | 1.0223 |
| 24 | 1.6426 | 1.5037 | 54 | 2.0133 | 1.2681 | 84 | 3.6519 | 1.0172 |
| 25 | 1.6490 | 1.4981 | 55 | 2.0347 | 1.2587 | 85 | 3.8317 | 1.0127 |
| 26 | 1.6557 | 1.4924 | 56 | 2.0571 | 1.2492 | 86 | 4.0528 | 1.0086 |
| 27 | 1.6627 | 1.4864 | 57 | 2.0804 | 1.2397 | 87 | 4.3387 | 1.0053 |
| 28 | 1.6701 | 1.4803 | 58 | 2.1047 | 1.2301 | 88 | 4.7427 | 1.0026 |
| 29 | 1.6777 | 1.4740 | 59 | 2.1300 | 1.2206 | 89 | 5.4349 | 1.0008 |
| 30 | 1.6858 | 1.4675 | 60 | 2.1565 | 1.2111 | 90 | $\infty$ | 1.0000 |

## TABLE
## 44

## INCOMPLETE ELLIPTIC INTEGRAL OF THE FIRST KIND

$$F(k, \phi) = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}, \quad k = \sin \psi$$

| $\phi$ \ $\psi$ | 0° | 10° | 20° | 30° | 40° | 50° | 60° | 70° | 80° | 90° |
|---|---|---|---|---|---|---|---|---|---|---|
| 0° | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 10° | 0.1745 | 0.1746 | 0.1746 | 0.1748 | 0.1749 | 0.1751 | 0.1752 | 0.1753 | 0.1754 | 0.1754 |
| 20° | 0.3491 | 0.3493 | 0.3499 | 0.3508 | 0.3520 | 0.3533 | 0.3545 | 0.3555 | 0.3561 | 0.3564 |
| 30° | 0.5236 | 0.5243 | 0.5263 | 0.5294 | 0.5334 | 0.5379 | 0.5422 | 0.5459 | 0.5484 | 0.5493 |
| 40° | 0.6981 | 0.6997 | 0.7043 | 0.7116 | 0.7213 | 0.7323 | 0.7436 | 0.7535 | 0.7604 | 0.7629 |
| 50° | 0.8727 | 0.8756 | 0.8842 | 0.8982 | 0.9173 | 0.9401 | 0.9647 | 0.9876 | 1.0044 | 1.0107 |
| 60° | 1.0472 | 1.0519 | 1.0660 | 1.0896 | 1.1226 | 1.1643 | 1.2126 | 1.2619 | 1.3014 | 1.3170 |
| 70° | 1.2217 | 1.2286 | 1.2495 | 1.2853 | 1.3372 | 1.4068 | 1.4944 | 1.5959 | 1.6918 | 1.7354 |
| 80° | 1.3963 | 1.4056 | 1.4344 | 1.4846 | 1.5597 | 1.6660 | 1.8125 | 2.0119 | 2.2653 | 2.4362 |
| 90° | 1.5708 | 1.5828 | 1.6200 | 1.6858 | 1.7868 | 1.9356 | 2.1565 | 2.5046 | 3.1534 | $\infty$ |

## TABLE
## 45

## INCOMPLETE ELLIPTIC INTEGRAL OF THE SECOND KIND

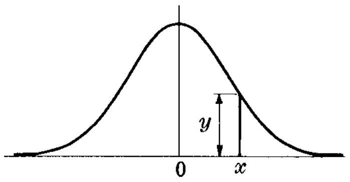
$$E(k, \phi) = \int_0^\phi \sqrt{1 - k^2 \sin^2 \theta}\, d\theta, \quad k = \sin \psi$$

| $\phi$ \ $\psi$ | 0° | 10° | 20° | 30° | 40° | 50° | 60° | 70° | 80° | 90° |
|---|---|---|---|---|---|---|---|---|---|---|
| 0° | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 10° | 0.1745 | 0.1745 | 0.1744 | 0.1743 | 0.1742 | 0.1740 | 0.1739 | 0.1738 | 0.1737 | 0.1736 |
| 20° | 0.3491 | 0.3489 | 0.3483 | 0.3473 | 0.3462 | 0.3450 | 0.3438 | 0.3429 | 0.3422 | 0.3420 |
| 30° | 0.5236 | 0.5229 | 0.5209 | 0.5179 | 0.5141 | 0.5100 | 0.5061 | 0.5029 | 0.5007 | 0.5000 |
| 40° | 0.6981 | 0.6966 | 0.6921 | 0.6851 | 0.6763 | 0.6667 | 0.6575 | 0.6497 | 0.6446 | 0.6428 |
| 50° | 0.8727 | 0.8698 | 0.8614 | 0.8483 | 0.8317 | 0.8134 | 0.7954 | 0.7801 | 0.7697 | 0.7660 |
| 60° | 1.0472 | 1.0426 | 1.0290 | 1.0076 | 0.9801 | 0.9493 | 0.9184 | 0.8914 | 0.8728 | 0.8660 |
| 70° | 1.2217 | 1.2149 | 1.1949 | 1.1632 | 1.1221 | 1.0750 | 1.0266 | 0.9830 | 0.9514 | 0.9397 |
| 80° | 1.3963 | 1.3870 | 1.3597 | 1.3161 | 1.2590 | 1.1926 | 1.1225 | 1.0565 | 1.0054 | 0.9848 |
| 90° | 1.5708 | 1.5589 | 1.5238 | 1.4675 | 1.3931 | 1.3055 | 1.2111 | 1.1184 | 1.0401 | 1.0000 |

## TABLE 46

## ORDINATES OF THE STANDARD NORMAL CURVE

$$y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$



| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | .3989 | .3989 | .3989 | .3988 | .3986 | .3984 | .3982 | .3980 | .3977 | .3973 |
| 0.1 | .3970 | .3965 | .3961 | .3956 | .3951 | .3945 | .3939 | .3932 | .3925 | .3918 |
| 0.2 | .3910 | .3902 | .3894 | .3885 | .3876 | .3867 | .3857 | .3847 | .3836 | .3825 |
| 0.3 | .3814 | .3802 | .3790 | .3778 | .3765 | .3752 | .3739 | .3725 | .3712 | .3697 |
| 0.4 | .3683 | .3668 | .3653 | .3637 | .3621 | .3605 | .3589 | .3572 | .3555 | .3538 |
| 0.5 | .3521 | .3503 | .3485 | .3467 | .3448 | .3429 | .3410 | .3391 | .3372 | .3352 |
| 0.6 | .3332 | .3312 | .3292 | .3271 | .3251 | .3230 | .3209 | .3187 | .3166 | .3144 |
| 0.7 | .3123 | .3101 | .3079 | .3056 | .3034 | .3011 | .2989 | .2966 | .2943 | .2920 |
| 0.8 | .2897 | .2874 | .2850 | .2827 | .2803 | .2780 | .2756 | .2732 | .2709 | .2685 |
| 0.9 | .2661 | .2637 | .2613 | .2589 | .2565 | .2541 | .2516 | .2492 | .2468 | .2444 |
| 1.0 | .2420 | .2396 | .2371 | .2347 | .2323 | .2299 | .2275 | .2251 | .2227 | .2203 |
| 1.1 | .2179 | .2155 | .2131 | .2107 | .2083 | .2059 | .2036 | .2012 | .1989 | .1965 |
| 1.2 | .1942 | .1919 | .1895 | .1872 | .1849 | .1826 | .1804 | .1781 | .1758 | .1736 |
| 1.3 | .1714 | .1691 | .1669 | .1647 | .1626 | .1604 | .1582 | .1561 | .1539 | .1518 |
| 1.4 | .1497 | .1476 | .1456 | .1435 | .1415 | .1394 | .1374 | .1354 | .1334 | .1315 |
| 1.5 | .1295 | .1276 | .1257 | .1238 | .1219 | .1200 | .1182 | .1163 | .1145 | .1127 |
| 1.6 | .1109 | .1092 | .1074 | .1057 | .1040 | .1023 | .1006 | .0989 | .0973 | .0957 |
| 1.7 | .0940 | .0925 | .0909 | .0893 | .0878 | .0863 | .0848 | .0833 | .0818 | .0804 |
| 1.8 | .0790 | .0775 | .0761 | .0748 | .0734 | .0721 | .0707 | .0694 | .0681 | .0669 |
| 1.9 | .0656 | .0644 | .0632 | .0620 | .0608 | .0596 | .0584 | .0573 | .0562 | .0551 |
| 2.0 | .0540 | .0529 | .0519 | .0508 | .0498 | .0488 | .0478 | .0468 | .0459 | .0449 |
| 2.1 | .0440 | .0431 | .0422 | .0413 | .0404 | .0396 | .0387 | .0379 | .0371 | .0363 |
| 2.2 | .0355 | .0347 | .0339 | .0332 | .0325 | .0317 | .0310 | .0303 | .0297 | .0290 |
| 2.3 | .0283 | .0277 | .0270 | .0264 | .0258 | .0252 | .0246 | .0241 | .0235 | .0229 |
| 2.4 | .0224 | .0219 | .0213 | .0208 | .0203 | .0198 | .0194 | .0189 | .0184 | .0180 |
| 2.5 | .0175 | .0171 | .0167 | .0163 | .0158 | .0154 | .0151 | .0147 | .0143 | .0139 |
| 2.6 | .0136 | .0132 | .0129 | .0126 | .0122 | .0119 | .0116 | .0113 | .0110 | .0107 |
| 2.7 | .0104 | .0101 | .0099 | .0096 | .0093 | .0091 | .0088 | .0086 | .0084 | .0081 |
| 2.8 | .0079 | .0077 | .0075 | .0073 | .0071 | .0069 | .0067 | .0065 | .0063 | .0061 |
| 2.9 | .0060 | .0058 | .0056 | .0055 | .0053 | .0051 | .0050 | .0048 | .0047 | .0046 |
| 3.0 | .0044 | .0043 | .0042 | .0040 | .0039 | .0038 | .0037 | .0036 | .0035 | .0034 |
| 3.1 | .0033 | .0032 | .0031 | .0030 | .0029 | .0028 | .0027 | .0026 | .0025 | .0025 |
| 3.2 | .0024 | .0023 | .0022 | .0022 | .0021 | .0020 | .0020 | .0019 | .0018 | .0018 |
| 3.3 | .0017 | .0017 | .0016 | .0016 | .0015 | .0015 | .0014 | .0014 | .0013 | .0013 |
| 3.4 | .0012 | .0012 | .0012 | .0011 | .0011 | .0010 | .0010 | .0010 | .0009 | .0009 |
| 3.5 | .0009 | .0008 | .0008 | .0008 | .0008 | .0007 | .0007 | .0007 | .0007 | .0006 |
| 3.6 | .0006 | .0006 | .0006 | .0005 | .0005 | .0005 | .0005 | .0005 | .0005 | .0004 |
| 3.7 | .0004 | .0004 | .0004 | .0004 | .0004 | .0004 | .0003 | .0003 | .0003 | .0003 |
| 3.8 | .0003 | .0003 | .0003 | .0003 | .0003 | .0002 | .0002 | .0002 | .0002 | .0002 |
| 3.9 | .0002 | .0002 | .0002 | .0002 | .0002 | .0002 | .0002 | .0002 | .0001 | .0001 |

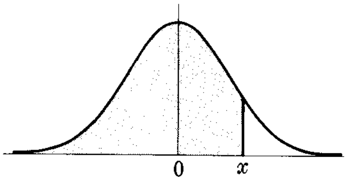
TABLE

47

## AREAS UNDER THE STANDARD NORMAL CURVE

from $-\infty$ to $x$

$$\operatorname{erf}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\,dt$$



| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | .5000 | .5040 | .5080 | .5120 | .5160 | .5199 | .5239 | .5279 | .5319 | .5359 |
| 0.1 | .5398 | .5438 | .5478 | .5517 | .5557 | .5596 | .5636 | .5675 | .5714 | .5754 |
| 0.2 | .5793 | .5832 | .5871 | .5910 | .5948 | .5987 | .6026 | .6064 | .6103 | .6141 |
| 0.3 | .6179 | .6217 | .6255 | .6293 | .6331 | .6368 | .6406 | .6443 | .6480 | .6517 |
| 0.4 | .6554 | .6591 | .6628 | .6664 | .6700 | .6736 | .6772 | .6808 | .6844 | .6879 |
| 0.5 | .6915 | .6950 | .6985 | .7019 | .7054 | .7088 | .7123 | .7157 | .7190 | .7224 |
| 0.6 | .7258 | .7291 | .7324 | .7357 | .7389 | .7422 | .7454 | .7486 | .7518 | .7549 |
| 0.7 | .7580 | .7612 | .7642 | .7673 | .7704 | .7734 | .7764 | .7794 | .7823 | .7852 |
| 0.8 | .7881 | .7910 | .7939 | .7967 | .7996 | .8023 | .8051 | .8078 | .8106 | .8133 |
| 0.9 | .8159 | .8186 | .8212 | .8238 | .8264 | .8289 | .8315 | .8340 | .8365 | .8389 |
| 1.0 | .8413 | .8438 | .8461 | .8485 | .8508 | .8531 | .8554 | .8577 | .8599 | .8621 |
| 1.1 | .8643 | .8665 | .8686 | .8708 | .8729 | .8749 | .8770 | .8790 | .8810 | .8830 |
| 1.2 | .8849 | .8869 | .8888 | .8907 | .8925 | .8944 | .8962 | .8980 | .8997 | .9015 |
| 1.3 | .9032 | .9049 | .9066 | .9082 | .9099 | .9115 | .9131 | .9147 | .9162 | .9177 |
| 1.4 | .9192 | .9207 | .9222 | .9236 | .9251 | .9265 | .9279 | .9292 | .9306 | .9319 |
| 1.5 | .9332 | .9345 | .9357 | .9370 | .9382 | .9394 | .9406 | .9418 | .9429 | .9441 |
| 1.6 | .9452 | .9463 | .9474 | .9484 | .9495 | .9505 | .9515 | .9525 | .9535 | .9545 |
| 1.7 | .9554 | .9564 | .9573 | .9582 | .9591 | .9599 | .9608 | .9616 | .9625 | .9633 |
| 1.8 | .9641 | .9649 | .9656 | .9664 | .9671 | .9678 | .9686 | .9693 | .9699 | .9706 |
| 1.9 | .9713 | .9719 | .9726 | .9732 | .9738 | .9744 | .9750 | .9756 | .9761 | .9767 |
| 2.0 | .9772 | .9778 | .9783 | .9788 | .9793 | .9798 | .9803 | .9808 | .9812 | .9817 |
| 2.1 | .9821 | .9826 | .9830 | .9834 | .9838 | .9842 | .9846 | .9850 | .9854 | .9857 |
| 2.2 | .9861 | .9864 | .9868 | .9871 | .9875 | .9878 | .9881 | .9884 | .9887 | .9890 |
| 2.3 | .9893 | .9896 | .9898 | .9901 | .9904 | .9906 | .9909 | .9911 | .9913 | .9916 |
| 2.4 | .9918 | .9920 | .9922 | .9925 | .9927 | .9929 | .9931 | .9932 | .9934 | .9936 |
| 2.5 | .9938 | .9940 | .9941 | .9943 | .9945 | .9946 | .9948 | .9949 | .9951 | .9952 |
| 2.6 | .9953 | .9955 | .9956 | .9957 | .9959 | .9960 | .9961 | .9962 | .9963 | .9964 |
| 2.7 | .9965 | .9966 | .9967 | .9968 | .9969 | .9970 | .9971 | .9972 | .9973 | .9974 |
| 2.8 | .9974 | .9975 | .9976 | .9977 | .9977 | .9978 | .9979 | .9979 | .9980 | .9981 |
| 2.9 | .9981 | .9982 | .9982 | .9983 | .9984 | .9984 | .9985 | .9985 | .9986 | .9986 |
| 3.0 | .9987 | .9987 | .9987 | .9988 | .9988 | .9989 | .9989 | .9989 | .9990 | .9990 |
| 3.1 | .9990 | .9991 | .9991 | .9991 | .9992 | .9992 | .9992 | .9992 | .9993 | .9993 |
| 3.2 | .9993 | .9993 | .9994 | .9994 | .9994 | .9994 | .9994 | .9995 | .9995 | .9995 |
| 3.3 | .9995 | .9995 | .9995 | .9996 | .9996 | .9996 | .9996 | .9996 | .9996 | .9997 |
| 3.4 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9998 |
| 3.5 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 |
| 3.6 | .9998 | .9998 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 |
| 3.7 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 |
| 3.8 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 |
| 3.9 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

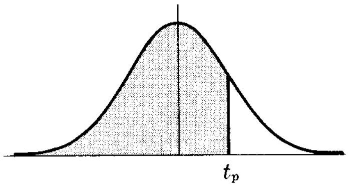
**TABLE**

**48**

## PERCENTILE VALUES ($t_p$) FOR STUDENT'S $t$ DISTRIBUTION

with $n$ degrees of freedom

(shaded area = $p$)



| $n$ | $t_{.995}$ | $t_{.99}$ | $t_{.975}$ | $t_{.95}$ | $t_{.90}$ | $t_{.80}$ | $t_{.75}$ | $t_{.70}$ | $t_{.60}$ | $t_{.55}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 63.66 | 31.82 | 12.71 | 6.31 | 3.08 | 1.376 | 1.000 | .727 | .325 | .158 |
| 2 | 9.92 | 6.96 | 4.30 | 2.92 | 1.89 | 1.061 | .816 | .617 | .289 | .142 |
| 3 | 5.84 | 4.54 | 3.18 | 2.35 | 1.64 | .978 | .765 | .584 | .277 | .137 |
| 4 | 4.60 | 3.75 | 2.78 | 2.13 | 1.53 | .941 | .741 | .569 | .271 | .134 |
| 5 | 4.03 | 3.36 | 2.57 | 2.02 | 1.48 | .920 | .727 | .559 | .267 | .132 |
| 6 | 3.71 | 3.14 | 2.45 | 1.94 | 1.44 | .906 | .718 | .553 | .265 | .131 |
| 7 | 3.50 | 3.00 | 2.36 | 1.90 | 1.42 | .896 | .711 | .549 | .263 | .130 |
| 8 | 3.36 | 2.90 | 2.31 | 1.86 | 1.40 | .889 | .706 | .546 | .262 | .130 |
| 9 | 3.25 | 2.82 | 2.26 | 1.83 | 1.38 | .883 | .703 | .543 | .261 | .129 |
| 10 | 3.17 | 2.76 | 2.23 | 1.81 | 1.37 | .879 | .700 | .542 | .260 | .129 |
| 11 | 3.11 | 2.72 | 2.20 | 1.80 | 1.36 | .876 | .697 | .540 | .260 | .129 |
| 12 | 3.06 | 2.68 | 2.18 | 1.78 | 1.36 | .873 | .695 | .539 | .259 | .128 |
| 13 | 3.01 | 2.65 | 2.16 | 1.77 | 1.35 | .870 | .694 | .538 | .259 | .128 |
| 14 | 2.98 | 2.62 | 2.14 | 1.76 | 1.34 | .868 | .692 | .537 | .258 | .128 |
| 15 | 2.95 | 2.60 | 2.13 | 1.75 | 1.34 | .866 | .691 | .536 | .258 | .128 |
| 16 | 2.92 | 2.58 | 2.12 | 1.75 | 1.34 | .865 | .690 | .535 | .258 | .128 |
| 17 | 2.90 | 2.57 | 2.11 | 1.74 | 1.33 | .863 | .689 | .534 | .257 | .128 |
| 18 | 2.88 | 2.55 | 2.10 | 1.73 | 1.33 | .862 | .688 | .534 | .257 | .127 |
| 19 | 2.86 | 2.54 | 2.09 | 1.73 | 1.33 | .861 | .688 | .533 | .257 | .127 |
| 20 | 2.84 | 2.53 | 2.09 | 1.72 | 1.32 | .860 | .687 | .533 | .257 | .127 |
| 21 | 2.83 | 2.52 | 2.08 | 1.72 | 1.32 | .859 | .686 | .532 | .257 | .127 |
| 22 | 2.82 | 2.51 | 2.07 | 1.72 | 1.32 | .858 | .686 | .532 | .256 | .127 |
| 23 | 2.81 | 2.50 | 2.07 | 1.71 | 1.32 | .858 | .685 | .532 | .256 | .127 |
| 24 | 2.80 | 2.49 | 2.06 | 1.71 | 1.32 | .857 | .685 | .531 | .256 | .127 |
| 25 | 2.79 | 2.48 | 2.06 | 1.71 | 1.32 | .856 | .684 | .531 | .256 | .127 |
| 26 | 2.78 | 2.48 | 2.06 | 1.71 | 1.32 | .856 | .684 | .531 | .256 | .127 |
| 27 | 2.77 | 2.47 | 2.05 | 1.70 | 1.31 | .855 | .684 | .531 | .256 | .127 |
| 28 | 2.76 | 2.47 | 2.05 | 1.70 | 1.31 | .855 | .683 | .530 | .256 | .127 |
| 29 | 2.76 | 2.46 | 2.04 | 1.70 | 1.31 | .854 | .683 | .530 | .256 | .127 |
| 30 | 2.75 | 2.46 | 2.04 | 1.70 | 1.31 | .854 | .683 | .530 | .256 | .127 |
| 40 | 2.70 | 2.42 | 2.02 | 1.68 | 1.30 | .851 | .681 | .529 | .255 | .126 |
| 60 | 2.66 | 2.39 | 2.00 | 1.67 | 1.30 | .848 | .679 | .527 | .254 | .126 |
| 120 | 2.62 | 2.36 | 1.98 | 1.66 | 1.29 | .845 | .677 | .526 | .254 | .126 |
| $\infty$ | 2.58 | 2.33 | 1.96 | 1.645 | 1.28 | .842 | .674 | .524 | .253 | .126 |

*Source:* R. A. Fisher and F. Yates, *Statistical Tables for Biological, Agricultural and Medical Research* (6th edition, 1963), Table III, Oliver and Boyd Ltd., Edinburgh, by permission of the authors and publishers.

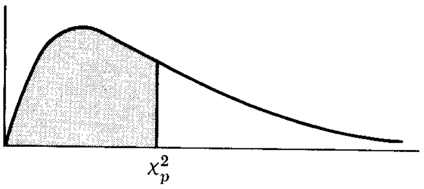
**TABLE**

# 49

## PERCENTILE VALUES ($\chi^2_p$) FOR THE CHI-SQUARE DISTRIBUTION

with $n$ degrees of freedom

(shaded area $= p$)



| $n$ | $\chi^2_{.995}$ | $\chi^2_{.99}$ | $\chi^2_{.975}$ | $\chi^2_{.95}$ | $\chi^2_{.90}$ | $\chi^2_{.75}$ | $\chi^2_{.50}$ | $\chi^2_{.25}$ | $\chi^2_{.10}$ | $\chi^2_{.05}$ | $\chi^2_{.025}$ | $\chi^2_{.01}$ | $\chi^2_{.005}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 7.88 | 6.63 | 5.02 | 3.84 | 2.71 | 1.32 | .455 | .102 | .0158 | .0039 | .0010 | .0002 | .0000 |
| 2 | 10.6 | 9.21 | 7.38 | 5.99 | 4.61 | 2.77 | 1.39 | .575 | .211 | .103 | .0506 | .0201 | .0100 |
| 3 | 12.8 | 11.3 | 9.35 | 7.81 | 6.25 | 4.11 | 2.37 | 1.21 | .584 | .352 | .216 | .115 | .072 |
| 4 | 14.9 | 13.3 | 11.1 | 9.49 | 7.78 | 5.39 | 3.36 | 1.92 | 1.06 | .711 | .484 | .297 | .207 |
| 5 | 16.7 | 15.1 | 12.8 | 11.1 | 9.24 | 6.63 | 4.35 | 2.67 | 1.61 | 1.15 | .831 | .554 | .412 |
| 6 | 18.5 | 16.8 | 14.4 | 12.6 | 10.6 | 7.84 | 5.35 | 3.45 | 2.20 | 1.64 | 1.24 | .872 | .676 |
| 7 | 20.3 | 18.5 | 16.0 | 14.1 | 12.0 | 9.04 | 6.35 | 4.25 | 2.83 | 2.17 | 1.69 | 1.24 | .989 |
| 8 | 22.0 | 20.1 | 17.5 | 15.5 | 13.4 | 10.2 | 7.34 | 5.07 | 3.49 | 2.73 | 2.18 | 1.65 | 1.34 |
| 9 | 23.6 | 21.7 | 19.0 | 16.9 | 14.7 | 11.4 | 8.34 | 5.90 | 4.17 | 3.33 | 2.70 | 2.09 | 1.73 |
| 10 | 25.2 | 23.2 | 20.5 | 18.3 | 16.0 | 12.5 | 9.34 | 6.74 | 4.87 | 3.94 | 3.25 | 2.56 | 2.16 |
| 11 | 26.8 | 24.7 | 21.9 | 19.7 | 17.3 | 13.7 | 10.3 | 7.58 | 5.58 | 4.57 | 3.82 | 3.05 | 2.60 |
| 12 | 28.3 | 26.2 | 23.3 | 21.0 | 18.5 | 14.8 | 11.3 | 8.44 | 6.30 | 5.23 | 4.40 | 3.57 | 3.07 |
| 13 | 29.8 | 27.7 | 24.7 | 22.4 | 19.8 | 16.0 | 12.3 | 9.30 | 7.04 | 5.89 | 5.01 | 4.11 | 3.57 |
| 14 | 31.3 | 29.1 | 26.1 | 23.7 | 21.1 | 17.1 | 13.3 | 10.2 | 7.79 | 6.57 | 5.63 | 4.66 | 4.07 |
| 15 | 32.8 | 30.6 | 27.5 | 25.0 | 22.3 | 18.2 | 14.3 | 11.0 | 8.55 | 7.26 | 6.26 | 5.23 | 4.60 |
| 16 | 34.3 | 32.0 | 28.8 | 26.3 | 23.5 | 19.4 | 15.3 | 11.9 | 9.31 | 7.96 | 6.91 | 5.81 | 5.14 |
| 17 | 35.7 | 33.4 | 30.2 | 27.6 | 24.8 | 20.5 | 16.3 | 12.8 | 10.1 | 8.67 | 7.56 | 6.41 | 5.70 |
| 18 | 37.2 | 34.8 | 31.5 | 28.9 | 26.0 | 21.6 | 17.3 | 13.7 | 10.9 | 9.39 | 8.23 | 7.01 | 6.26 |
| 19 | 38.6 | 36.2 | 32.9 | 30.1 | 27.2 | 22.7 | 18.3 | 14.6 | 11.7 | 10.1 | 8.91 | 7.63 | 6.84 |
| 20 | 40.0 | 37.6 | 34.2 | 31.4 | 28.4 | 23.8 | 19.3 | 15.5 | 12.4 | 10.9 | 9.59 | 8.26 | 7.43 |
| 21 | 41.4 | 38.9 | 35.5 | 32.7 | 29.6 | 24.9 | 20.3 | 16.3 | 13.2 | 11.6 | 10.3 | 8.90 | 8.03 |
| 22 | 42.8 | 40.3 | 36.8 | 33.9 | 30.8 | 26.0 | 21.3 | 17.2 | 14.0 | 12.3 | 11.0 | 9.54 | 8.64 |
| 23 | 44.2 | 41.6 | 38.1 | 35.2 | 32.0 | 27.1 | 22.3 | 18.1 | 14.8 | 13.1 | 11.7 | 10.2 | 9.26 |
| 24 | 45.6 | 43.0 | 39.4 | 36.4 | 33.2 | 28.2 | 23.3 | 19.0 | 15.7 | 13.8 | 12.4 | 10.9 | 9.89 |
| 25 | 46.9 | 44.3 | 40.6 | 37.7 | 34.4 | 29.3 | 24.3 | 19.9 | 16.5 | 14.6 | 13.1 | 11.5 | 10.5 |
| 26 | 48.3 | 45.6 | 41.9 | 38.9 | 35.6 | 30.4 | 25.3 | 20.8 | 17.3 | 15.4 | 13.8 | 12.2 | 11.2 |
| 27 | 49.6 | 47.0 | 43.2 | 40.1 | 36.7 | 31.5 | 26.3 | 21.7 | 18.1 | 16.2 | 14.6 | 12.9 | 11.8 |
| 28 | 51.0 | 48.3 | 44.5 | 41.3 | 37.9 | 32.6 | 27.3 | 22.7 | 18.9 | 16.9 | 15.3 | 13.6 | 12.5 |
| 29 | 52.3 | 49.6 | 45.7 | 42.6 | 39.1 | 33.7 | 28.3 | 23.6 | 19.8 | 17.7 | 16.0 | 14.3 | 13.1 |
| 30 | 53.7 | 50.9 | 47.0 | 43.8 | 40.3 | 34.8 | 29.3 | 24.5 | 20.6 | 18.5 | 16.8 | 15.0 | 13.8 |
| 40 | 66.8 | 63.7 | 59.3 | 55.8 | 51.8 | 45.6 | 39.3 | 33.7 | 29.1 | 26.5 | 24.4 | 22.2 | 20.7 |
| 50 | 79.5 | 76.2 | 71.4 | 67.5 | 63.2 | 56.3 | 49.3 | 42.9 | 37.7 | 34.8 | 32.4 | 29.7 | 28.0 |
| 60 | 92.0 | 88.4 | 83.3 | 79.1 | 74.4 | 67.0 | 59.3 | 52.3 | 46.5 | 43.2 | 40.5 | 37.5 | 35.5 |
| 70 | 104.2 | 100.4 | 95.0 | 90.5 | 85.5 | 77.6 | 69.3 | 61.7 | 55.3 | 51.7 | 48.8 | 45.4 | 43.3 |
| 80 | 116.3 | 112.3 | 106.6 | 101.9 | 96.6 | 88.1 | 79.3 | 71.1 | 64.3 | 60.4 | 57.2 | 53.5 | 51.2 |
| 90 | 128.3 | 124.1 | 118.1 | 113.1 | 107.6 | 98.6 | 89.3 | 80.6 | 73.3 | 69.1 | 65.6 | 61.8 | 59.2 |
| 100 | 140.2 | 135.8 | 129.6 | 124.3 | 118.5 | 109.1 | 99.3 | 90.1 | 82.4 | 77.9 | 74.2 | 70.1 | 67.3 |

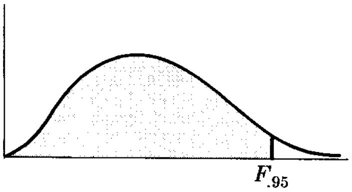
*Source:* Catherine M. Thompson, *Table of percentage points of the $\chi^2$ distribution,* Biometrika, Vol. 32 (1941), by permission of the author and publisher.

**TABLE**

**50**

## 95th PERCENTILE VALUES FOR THE $F$ DISTRIBUTION

$n_1$ = degrees of freedom for numerator
$n_2$ = degrees of freedom for denominator
(shaded area = .95)



| $n_2$ \ $n_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 12 | 16 | 20 | 30 | 40 | 50 | 100 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 161.4 | 199.5 | 215.7 | 224.6 | 230.2 | 234.0 | 238.9 | 243.9 | 246.3 | 248.0 | 250.1 | 251.1 | 252.2 | 253.0 | 254.3 |
| 2 | 18.51 | 19.00 | 19.16 | 19.25 | 19.30 | 19.33 | 19.37 | 19.41 | 19.43 | 19.45 | 19.46 | 19.46 | 19.47 | 19.49 | 19.50 |
| 3 | 10.13 | 9.55 | 9.28 | 9.12 | 9.01 | 8.94 | 8.85 | 8.74 | 8.69 | 8.66 | 8.62 | 8.60 | 8.58 | 8.56 | 8.53 |
| 4 | 7.71 | 6.94 | 6.59 | 6.39 | 6.26 | 6.16 | 6.04 | 5.91 | 5.84 | 5.80 | 5.75 | 5.71 | 5.70 | 5.66 | 5.63 |
| 5 | 6.61 | 5.79 | 5.41 | 5.19 | 5.05 | 4.95 | 4.82 | 4.68 | 4.60 | 4.56 | 4.50 | 4.46 | 4.44 | 4.40 | 4.36 |
| 6 | 5.99 | 5.14 | 4.76 | 4.53 | 4.39 | 4.28 | 4.15 | 4.00 | 3.92 | 3.87 | 3.81 | 3.77 | 3.75 | 3.71 | 3.67 |
| 7 | 5.59 | 4.74 | 4.35 | 4.12 | 3.97 | 3.87 | 3.73 | 3.57 | 3.49 | 3.44 | 3.38 | 3.34 | 3.32 | 3.28 | 3.23 |
| 8 | 5.32 | 4.46 | 4.07 | 3.84 | 3.69 | 3.58 | 3.44 | 3.28 | 3.20 | 3.15 | 3.08 | 3.05 | 3.03 | 2.98 | 2.93 |
| 9 | 5.12 | 4.26 | 3.86 | 3.63 | 3.48 | 3.37 | 3.23 | 3.07 | 2.98 | 2.93 | 2.86 | 2.82 | 2.80 | 2.76 | 2.71 |
| 10 | 4.96 | 4.10 | 3.71 | 3.48 | 3.33 | 3.22 | 3.07 | 2.91 | 2.82 | 2.77 | 2.70 | 2.67 | 2.64 | 2.59 | 2.54 |
| 11 | 4.84 | 3.98 | 3.59 | 3.36 | 3.20 | 3.09 | 2.95 | 2.79 | 2.70 | 2.65 | 2.57 | 2.53 | 2.50 | 2.45 | 2.40 |
| 12 | 4.75 | 3.89 | 3.49 | 3.26 | 3.11 | 3.00 | 2.85 | 2.69 | 2.60 | 2.54 | 2.46 | 2.42 | 2.40 | 2.35 | 2.30 |
| 13 | 4.67 | 3.81 | 3.41 | 3.18 | 3.03 | 2.92 | 2.77 | 2.60 | 2.51 | 2.46 | 2.38 | 2.34 | 2.32 | 2.26 | 2.21 |
| 14 | 4.60 | 3.74 | 3.34 | 3.11 | 2.96 | 2.85 | 2.70 | 2.53 | 2.44 | 2.39 | 2.31 | 2.27 | 2.24 | 2.19 | 2.13 |
| 15 | 4.54 | 3.68 | 3.29 | 3.06 | 2.90 | 2.79 | 2.64 | 2.48 | 2.39 | 2.33 | 2.25 | 2.21 | 2.18 | 2.12 | 2.07 |
| 16 | 4.49 | 3.63 | 3.24 | 3.01 | 2.85 | 2.74 | 2.59 | 2.42 | 2.33 | 2.28 | 2.20 | 2.16 | 2.13 | 2.07 | 2.01 |
| 17 | 4.45 | 3.59 | 3.20 | 2.96 | 2.81 | 2.70 | 2.55 | 2.38 | 2.29 | 2.23 | 2.15 | 2.11 | 2.08 | 2.02 | 1.96 |
| 18 | 4.41 | 3.55 | 3.16 | 2.93 | 2.77 | 2.66 | 2.51 | 2.34 | 2.25 | 2.19 | 2.11 | 2.07 | 2.04 | 1.98 | 1.92 |
| 19 | 4.38 | 3.52 | 3.13 | 2.90 | 2.74 | 2.63 | 2.48 | 2.31 | 2.21 | 2.15 | 2.07 | 2.02 | 2.00 | 1.94 | 1.88 |
| 20 | 4.35 | 3.49 | 3.10 | 2.87 | 2.71 | 2.60 | 2.45 | 2.28 | 2.18 | 2.12 | 2.04 | 1.99 | 1.96 | 1.90 | 1.84 |
| 22 | 4.30 | 3.44 | 3.05 | 2.82 | 2.66 | 2.55 | 2.40 | 2.23 | 2.13 | 2.07 | 1.98 | 1.93 | 1.91 | 1.84 | 1.78 |
| 24 | 4.26 | 3.40 | 3.01 | 2.78 | 2.62 | 2.51 | 2.36 | 2.18 | 2.09 | 2.03 | 1.94 | 1.89 | 1.86 | 1.80 | 1.73 |
| 26 | 4.23 | 3.37 | 2.98 | 2.74 | 2.59 | 2.47 | 2.32 | 2.15 | 2.05 | 1.99 | 1.90 | 1.85 | 1.82 | 1.76 | 1.69 |
| 28 | 4.20 | 3.34 | 2.95 | 2.71 | 2.56 | 2.45 | 2.29 | 2.12 | 2.02 | 1.96 | 1.87 | 1.81 | 1.78 | 1.72 | 1.65 |
| 30 | 4.17 | 3.32 | 2.92 | 2.69 | 2.53 | 2.42 | 2.27 | 2.09 | 1.99 | 1.93 | 1.84 | 1.79 | 1.76 | 1.69 | 1.62 |
| 40 | 4.08 | 3.23 | 2.84 | 2.61 | 2.45 | 2.34 | 2.18 | 2.00 | 1.90 | 1.84 | 1.74 | 1.69 | 1.66 | 1.59 | 1.51 |
| 50 | 4.03 | 3.18 | 2.79 | 2.56 | 2.40 | 2.29 | 2.13 | 1.95 | 1.85 | 1.78 | 1.69 | 1.63 | 1.60 | 1.52 | 1.44 |
| 60 | 4.00 | 3.15 | 2.76 | 2.53 | 2.37 | 2.25 | 2.10 | 1.92 | 1.81 | 1.75 | 1.65 | 1.59 | 1.56 | 1.48 | 1.39 |
| 70 | 3.98 | 3.13 | 2.74 | 2.50 | 2.35 | 2.23 | 2.07 | 1.89 | 1.79 | 1.72 | 1.62 | 1.56 | 1.53 | 1.45 | 1.35 |
| 80 | 3.96 | 3.11 | 2.72 | 2.48 | 2.33 | 2.21 | 2.05 | 1.88 | 1.77 | 1.70 | 1.60 | 1.54 | 1.51 | 1.42 | 1.32 |
| 100 | 3.94 | 3.09 | 2.70 | 2.46 | 2.30 | 2.19 | 2.03 | 1.85 | 1.75 | 1.68 | 1.57 | 1.51 | 1.48 | 1.39 | 1.28 |
| 150 | 3.91 | 3.06 | 2.67 | 2.43 | 2.27 | 2.16 | 2.00 | 1.82 | 1.71 | 1.64 | 1.54 | 1.47 | 1.44 | 1.34 | 1.22 |
| 200 | 3.89 | 3.04 | 2.65 | 2.41 | 2.26 | 2.14 | 1.98 | 1.80 | 1.69 | 1.62 | 1.52 | 1.45 | 1.42 | 1.32 | 1.19 |
| 400 | 3.86 | 3.02 | 2.62 | 2.39 | 2.23 | 2.12 | 1.96 | 1.78 | 1.67 | 1.60 | 1.49 | 1.42 | 1.38 | 1.28 | 1.13 |
| ∞ | 3.84 | 2.99 | 2.60 | 2.37 | 2.21 | 2.09 | 1.94 | 1.75 | 1.64 | 1.57 | 1.46 | 1.40 | 1.32 | 1.24 | 1.00 |

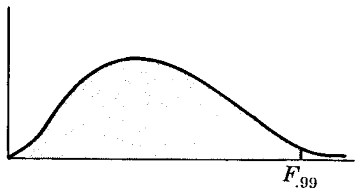
*Source:* G. W. Snedecor and W. G. Cochran, *Statistical Methods* (6th edition, 1967), Iowa State University Press, Ames, Iowa, by permission of the authors and publisher.

**TABLE**

# 51

## 99th PERCENTILE VALUES FOR THE $F$ DISTRIBUTION

$n_1$ = degrees of freedom for numerator
$n_2$ = degrees of freedom for denominator
(shaded area = .99)



| $n_1$ $n_2$ | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 12 | 16 | 20 | 30 | 40 | 50 | 100 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 4052 | 4999 | 5403 | 5625 | 5764 | 5859 | 5981 | 6106 | 6169 | 6208 | 6258 | 6286 | 6302 | 6334 | 6366 |
| 2 | 98.49 | 99.01 | 99.17 | 99.25 | 99.30 | 99.33 | 99.36 | 99.42 | 99.44 | 99.45 | 99.47 | 99.48 | 99.48 | 99.49 | 99.50 |
| 3 | 34.12 | 30.81 | 29.46 | 28.71 | 28.24 | 27.41 | 27.49 | 27.05 | 28.63 | 26.69 | 26.50 | 26.41 | 26.35 | 26.23 | 26.12 |
| 4 | 21.20 | 18.00 | 16.69 | 15.98 | 15.52 | 15.21 | 14.80 | 14.37 | 14.15 | 14.02 | 13.83 | 13.74 | 13.69 | 13.57 | 13.46 |
| 5 | 16.26 | 13.27 | 12.06 | 11.39 | 10.97 | 10.67 | 10.27 | 9.89 | 9.68 | 9.55 | 9.38 | 9.29 | 9.24 | 9.13 | 9.02 |
| 6 | 13.74 | 10.92 | 9.78 | 9.15 | 8.75 | 8.47 | 8.10 | 7.72 | 7.52 | 7.39 | 7.23 | 7.14 | 7.09 | 6.99 | 6.88 |
| 7 | 12.25 | 9.55 | 8.45 | 7.85 | 7.46 | 7.19 | 6.84 | 6.47 | 6.27 | 6.15 | 5.98 | 5.90 | 5.85 | 5.75 | 5.65 |
| 8 | 11.26 | 8.65 | 7.59 | 7.01 | 6.63 | 6.37 | 6.03 | 5.67 | 5.48 | 5.36 | 5.20 | 5.11 | 5.06 | 4.96 | 4.86 |
| 9 | 10.56 | 8.02 | 6.99 | 6.42 | 6.06 | 5.80 | 5.47 | 5.11 | 4.92 | 4.80 | 4.64 | 4.56 | 4.51 | 4.41 | 4.31 |
| 10 | 10.04 | 7.56 | 6.55 | 5.99 | 5.64 | 5.39 | 5.06 | 4.71 | 4.52 | 4.41 | 4.25 | 4.17 | 4.12 | 4.01 | 3.91 |
| 11 | 9.05 | 7.20 | 6.22 | 5.67 | 5.32 | 5.07 | 4.74 | 4.40 | 4.21 | 4.10 | 3.94 | 3.86 | 3.80 | 3.70 | 3.60 |
| 12 | 9.33 | 6.93 | 5.95 | 5.41 | 5.06 | 4.82 | 4.50 | 4.16 | 3.98 | 3.86 | 3.70 | 3.61 | 3.56 | 3.46 | 3.36 |
| 13 | 9.07 | 6.70 | 5.74 | 5.20 | 4.86 | 4.62 | 4.30 | 3.96 | 3.78 | 3.67 | 3.51 | 3.42 | 3.37 | 3.27 | 3.16 |
| 14 | 8.86 | 6.51 | 5.56 | 5.03 | 4.69 | 4.46 | 4.14 | 3.80 | 3.62 | 3.51 | 3.34 | 3.26 | 3.21 | 3.11 | 3.00 |
| 15 | 8.68 | 6.36 | 5.42 | 4.89 | 4.56 | 4.32 | 4.00 | 3.67 | 3.48 | 3.36 | 3.20 | 3.12 | 3.07 | 2.97 | 2.87 |
| 16 | 8.53 | 6.23 | 5.29 | 4.77 | 4.44 | 4.20 | 3.89 | 3.55 | 3.37 | 3.25 | 3.10 | 3.01 | 2.96 | 2.86 | 2.75 |
| 17 | 8.40 | 6.11 | 5.18 | 4.67 | 4.34 | 4.10 | 3.79 | 3.45 | 3.27 | 3.16 | 3.00 | 2.92 | 2.86 | 2.76 | 2.65 |
| 18 | 8.28 | 6.01 | 5.09 | 4.58 | 4.25 | 4.01 | 3.71 | 3.37 | 3.19 | 3.07 | 2.91 | 2.83 | 2.78 | 2.68 | 2.57 |
| 19 | 8.18 | 5.93 | 5.01 | 4.50 | 4.17 | 3.94 | 3.63 | 3.30 | 3.12 | 3.00 | 2.84 | 2.76 | 2.70 | 2.60 | 2.49 |
| 20 | 8.10 | 5.85 | 4.94 | 4.43 | 4.10 | 3.87 | 3.56 | 3.23 | 3.05 | 2.94 | 2.77 | 2.69 | 2.63 | 2.53 | 2.42 |
| 22 | 7.94 | 5.72 | 4.82 | 4.31 | 3.99 | 3.76 | 3.45 | 3.12 | 2.94 | 2.83 | 2.67 | 2.58 | 2.53 | 2.42 | 2.31 |
| 24 | 7.82 | 5.61 | 4.72 | 4.22 | 3.90 | 3.67 | 3.36 | 3.03 | 2.85 | 2.74 | 2.58 | 2.49 | 2.44 | 2.33 | 2.21 |
| 26 | 7.72 | 5.53 | 4.64 | 4.14 | 3.82 | 3.59 | 3.29 | 2.96 | 2.77 | 2.66 | 2.50 | 2.41 | 2.36 | 2.25 | 2.13 |
| 28 | 7.64 | 5.45 | 4.57 | 4.07 | 3.76 | 3.53 | 3.23 | 2.90 | 2.71 | 2.60 | 2.44 | 2.35 | 2.30 | 2.18 | 2.06 |
| 30 | 7.56 | 5.39 | 4.51 | 4.02 | 3.70 | 3.47 | 3.17 | 2.84 | 2.66 | 2.55 | 2.38 | 2.29 | 2.24 | 2.13 | 2.01 |
| 40 | 7.31 | 5.18 | 4.31 | 3.83 | 3.51 | 3.29 | 2.99 | 2.66 | 2.49 | 2.37 | 2.20 | 2.11 | 2.05 | 1.94 | 1.81 |
| 50 | 7.17 | 5.06 | 4.20 | 3.72 | 3.41 | 3.18 | 2.88 | 2.56 | 2.39 | 2.26 | 2.10 | 2.00 | 1.94 | 1.82 | 1.68 |
| 60 | 7.08 | 4.98 | 4.13 | 3.65 | 3.34 | 3.12 | 2.82 | 2.50 | 2.32 | 2.20 | 2.03 | 1.93 | 1.87 | 1.74 | 1.60 |
| 70 | 7.01 | 4.92 | 4.08 | 3.60 | 3.29 | 3.07 | 2.77 | 2.45 | 2.28 | 2.15 | 1.98 | 1.88 | 1.82 | 1.69 | 1.53 |
| 80 | 6.96 | 4.88 | 4.04 | 3.56 | 3.25 | 3.04 | 2.74 | 2.41 | 2.24 | 2.11 | 1.94 | 1.84 | 1.78 | 1.65 | 1.49 |
| 100 | 6.90 | 4.82 | 3.98 | 3.51 | 3.20 | 2.99 | 2.69 | 2.36 | 2.19 | 2.06 | 1.89 | 1.79 | 1.73 | 1.59 | 1.43 |
| 150 | 6.81 | 4.75 | 3.91 | 3.44 | 3.14 | 2.92 | 2.62 | 2.30 | 2.12 | 2.00 | 1.83 | 1.72 | 1.66 | 1.51 | 1.33 |
| 200 | 6.76 | 4.71 | 3.88 | 3.41 | 3.11 | 2.90 | 2.60 | 2.28 | 2.09 | 1.97 | 1.79 | 1.69 | 1.62 | 1.48 | 1.28 |
| 400 | 6.70 | 4.66 | 3.83 | 3.36 | 3.06 | 2.85 | 2.55 | 2.23 | 2.04 | 1.92 | 1.74 | 1.64 | 1.57 | 1.42 | 1.19 |
| ∞ | 6.64 | 4.60 | 3.78 | 3.32 | 3.02 | 2.80 | 2.51 | 2.18 | 1.99 | 1.87 | 1.69 | 1.59 | 1.52 | 1.36 | 1.00 |

*Source:* G. W. Snedecor and W. G. Cochran, *Statistical Methods* (6th edition, 1967), Iowa State University Press, Ames, Iowa, by permission of the authors and publisher.

TABLE

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## RANDOM NUMBERS

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 51772 | 74640 | 42331 | 29044 | 46621 | 62898 | 93582 | 04186 | 19640 | 87056 |
| 24033 | 23491 | 83587 | 06568 | 21960 | 21387 | 76105 | 10863 | 97453 | 90581 |
| 45939 | 60173 | 52078 | 25424 | 11645 | 55870 | 56974 | 37428 | 93507 | 94271 |
| 30586 | 02133 | 75797 | 45406 | 31041 | 86707 | 12973 | 17169 | 88116 | 42187 |
| 03585 | 79353 | 81938 | 82322 | 96799 | 85659 | 36081 | 50884 | 14070 | 74950 |
| 64937 | 03355 | 95863 | 20790 | 65304 | 55189 | 00745 | 65253 | 11822 | 15804 |
| 15630 | 64759 | 51135 | 98527 | 62586 | 41889 | 25439 | 88036 | 24034 | 67283 |
| 09448 | 56301 | 57683 | 30277 | 94623 | 85418 | 68829 | 06652 | 41982 | 49159 |
| 21631 | 91157 | 77331 | 60710 | 52290 | 16835 | 48653 | 71590 | 16159 | 14676 |
| 91097 | 17480 | 29414 | 06829 | 87843 | 28195 | 27279 | 47152 | 35683 | 47280 |
| 50532 | 25496 | 95652 | 42457 | 73547 | 76552 | 50020 | 24819 | 52984 | 76168 |
| 07136 | 40876 | 79971 | 54195 | 25708 | 51817 | 36732 | 72484 | 94923 | 75936 |
| 27989 | 64728 | 10744 | 08396 | 56242 | 90985 | 28868 | 99431 | 50995 | 20507 |
| 85184 | 73949 | 36601 | 46253 | 00477 | 25234 | 09908 | 36574 | 72139 | 70185 |
| 54398 | 21154 | 97810 | 36764 | 32869 | 11785 | 55261 | 59009 | 38714 | 38723 |
| 65544 | 34371 | 09591 | 07839 | 58892 | 92843 | 72828 | 91341 | 84821 | 63886 |
| 08263 | 65952 | 85762 | 64236 | 39238 | 18776 | 84303 | 99247 | 46149 | 03229 |
| 39817 | 67906 | 48236 | 16057 | 81812 | 15815 | 63700 | 85915 | 19219 | 45943 |
| 62257 | 04077 | 79443 | 95203 | 02479 | 30763 | 92486 | 54083 | 23631 | 05825 |
| 53298 | 90276 | 62545 | 21944 | 16530 | 03878 | 07516 | 95715 | 02526 | 33537 |

54.581

A 2.85

# Index of Special Symbols and Notations

The following list shows special symbols and notations used in this book together with pages on which they are defined or first appear. Cases where a symbol has more than one meaning will be clear from the context.

## *Symbols*

| | |
|---|---|
| $\mathrm{Ber}_n(x)$, $\mathrm{Bei}_n(x)$ | 140 |
| $B(m, n)$ | beta function, 103 |
| $B_n$ | Bernoulli numbers, 114 |
| $C(x)$ | Fresnel cosine integral, 184 |
| $Ci(x)$ | cosine integral, 184 |
| $e$ | natural base of logarithms, 1 |
| $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ | unit vectors in curvilinear coordinates, 124 |
| $\mathrm{erf}\,(x)$ | error function, 183 |
| $\mathrm{erfc}\,(x)$ | complementary error function, 183 |
| $E = E(k, \pi/2)$ | complete elliptic integral of second kind, 179 |
| $E(k, \phi)$ | incomplete elliptic integral of second kind, 179 |
| $Ei(x)$ | exponential integral, 183 |
| $E_n$ | Euler numbers, 114 |
| $F(a, b; c; x)$ | hypergeometric function, 160 |
| $F(k, \phi)$ | incomplete elliptic integral of first kind, 179 |
| $\mathcal{F}, \mathcal{F}^{-1}$ | Fourier transform and inverse Fourier transform, 175, 176 |
| $h_1, h_2, h_3$ | scale factors in curvilinear coordinates, 124 |
| $H_n(x)$ | Hermite polynomials, 151 |
| $H_n^{(1)}(x)$, $H_n^{(2)}(x)$ | Hankel functions of first and second kind, 138 |
| $i$ | imaginary unit, 21 |
| $\mathbf{i}, \mathbf{j}, \mathbf{k}$ | unit vectors in rectangular coordinates, 117 |
| $I_n(x)$ | modified Bessel function of first kind, 138 |
| $J_n(x)$ | Bessel function of first kind, 136 |
| $K = F(k, \pi/2)$ | complete elliptic integral of first kind, 179 |
| $\mathrm{Ker}_n(x)$, $\mathrm{Kei}_n(x)$ | 140 |
| $K_n(x)$ | modified Bessel function of second kind, 139 |
| $\ln x$ or $\log_e x$ | natural logarithm of $x$, 24 |
| $\log x$ or $\log_{10} x$ | common logarithm of $x$, 23 |
| $L_n(x)$ | Laguerre polynomials, 153 |
| $L_n^m(x)$ | associated Laguerre polynomials, 155 |
| $\mathcal{L}, \mathcal{L}^{-1}$ | Laplace transform and inverse Laplace transform, 161 |
| $P_n(x)$ | Legendre polynomials, 146 |
| $P_n^m(x)$ | associated Legendre functions of first kind, 149 |
| $Q_n(x)$ | Legendre functions of second kind, 148 |
| $Q_n^m(x)$ | associated Legendre functions of second kind, 150 |
| $r$ | cylindrical coordinate, 49 |
| $r$ | polar coordinate, 22, 36 |
| $r$ | spherical coordinate, 50 |
| $S(x)$ | Fresnel sine integral, 184 |
| $Si(x)$ | sine integral, 183 |
| $T_n(x)$ | Chebyshev polynomials of first kind, 157 |
| $U_n(x)$ | Chebyshev polynomials of second kind, 158 |
| $Y_n(x)$ | Bessel function of second kind, 136 |

## *Greek Symbols*

| | |
|---|---|
| $\gamma$ | Euler's constant, 1 |
| $\Gamma(x)$ | gamma function, 1, 101 |
| $\zeta(x)$ | Riemann zeta function, 184 |
| $\theta$ | cylindrical coordinate, 49 |
| $\theta$ | polar coordinate, 22, 36 |
| $\theta$ | spherical coordinate, 50 |
| $\pi$ | 1 |
| $\phi$ | spherical coordinate, 50 |
| $\Phi(p)$ | the sum $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p}$, $\Phi(0) = 0$, 137 |
| $\Phi(x)$ | probability distribution function, 189 |

## *Notations*

| | |
|---|---|
| $A = B$ | $A$ equals $B$ or $A$ is equal to $B$ |
| $A > B$ | $A$ is greater than $B$ [or $B$ is less than $A$] |
| $A < B$ | $A$ is less than $B$ [or $B$ is greater than $A$] |
| $A \geqq B$ | $A$ is greater than or equal to $B$ |
| $A \leqq B$ | $A$ is less than or equal to $B$ |
| $A \approx B$ | $A$ is approximately equal to $B$ |
| $A \sim B$ | $A$ is asymptotic to $B$ or $A/B$ approaches 1, 102 |
| $\lvert A \rvert$ | absolute value of $A = \begin{cases} A & \text{if } A \geqq 0 \\ -A & \text{if } A \leqq 0 \end{cases}$ |
| $n!$ | factorial $n$, 3 |
| $\binom{n}{k}$ | binomial coefficients, 3 |
| $\left.\begin{array}{r} y' = \frac{dy}{dx} = f'(x), \\ y'' = \frac{d^2y}{dx^2} = f''(x), \text{ etc.} \end{array}\right\}$ | derivatives of $y$ or $f(x)$ with respect to $x$, 53, 55 |
| $D^p = \frac{d^p}{dx^p}$ | $p$th derivative with respect to $x$, 55 |
| $dy$ | differential of $y$, 55 |
| $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x\, \partial y}$, etc. | partial derivatives, 56 |
| $\frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)}$ | Jacobian, 125 |
| $\int f(x)\, dx$ | indefinite integral, 57 |
| $\int_a^b f(x)\, dx$ | definite integral, 94 |
| $\int_C \mathbf{A} \cdot d\mathbf{r}$ | line integral of **A** along $C$, 121 |
| $\mathbf{A} \cdot \mathbf{B}$ | dot product of **A** and **B**, 117 |
| $\mathbf{A} \times \mathbf{B}$ | cross product of **A** and **B**, 118 |
| $\nabla$ | del operator, 119 |
| $\nabla^2 = \nabla \cdot \nabla$ | Laplacian operator, 120 |
| $\nabla^4 = \nabla^2(\nabla^2)$ | biharmonic operator, 120 |

# INDEX

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